



**Misir Mardanov
Varga Kalantarov
Elkhan Sabziev (Eds.)**



**13th International Conference on
Modern Problems of
Mathematics and Mechanics**

ABSTRACTS

**September 9-12, 2026
Baku, Azerbaijan**

Institute of Mathematics (Azerbaijan)

Baku State University (Azerbaijan)

**National University of Uzbekistan named after Mirzo Ulugbek
(Uzbekistan)**

Vasyl Stefanyk Carpathian National University (Ukraine)

Institute of Information Technology (Azerbaijan)

A B S T R A C T S

**OF THE 13th INTERNATIONAL CONFERENCE
“MODERN PROBLEMS OF
MATHEMATICS AND MECHANICS”**

**September 9-12, 2026
Baku, Azerbaijan**

Organizing Committee:

Chair: Elkhon Sabziev (Azerbaijan)

Co-Chair: Fahreddin Abdullayev (Türkiye)

Executive Secretary: Tofiq Babayev (Azerbaijan)

Members:

Abbasov Eldar (Azerbaijan), Akhundov Adalat (Azerbaijan), Aliyeva Aygun (Azerbaijan), Aripov Mersaid (Uzbekistan), Bagirov Emin (Azerbaijan), Azimov Rustam (Azerbaijan), Dmytryshyn Roman (Ukraine), Guliyev Hamlet (Azerbaijan), Karimov Etiram (Azerbaijan), Kocinac Ljubisa (Serbia), Mamedov Eminaga (Azerbaijan), Mamedov Yusif (Azerbaijan), Rassulova Nazila (Azerbaijan), Talybly Latif (Azerbaijan), Zikirov Obidjon (Uzbekistan)

Program Committee:

Chair: Misir Mardanov (Azerbaijan)

Co-Chair: Varga Kalantarov (Türkiye), Inom Madjidov (Uzbekistan)

Members:

Abbas Mujahid (Africa), Aida-zade Kamil (Azerbaijan), Aliev Araz (Azerbaijan), Aliguliyev Rasim (Azerbaijan), Aliyev Ziyatkhan (Azerbaijan), Andriyevskyy Volodymyr (USA), Ayupov Shavkat (Uzbekistan), Bandaliyev Rovshan (Azerbaijan), Bilalov Bilal (Azerbaijan), Blatt Hans Peter (Germany), Daşdemir Ahmet (Türkiye), Efendiev Yalchin (USA), Garayev Mübariz (Saudi Arabia), Gasimov Yusif (Azerbaijan), Gauthier Paul (Canada), Goginava Ushangi (Abu Dhabi, UAE), Golberg Anatoly (Israel), Guliyev Vagif (Azerbaijan), Imashkyzy Meerim (Kyrgyzstan), Ismailov Vugar (Azerbaijan), Kalimoldaev Maksat (Kazakhstan), Kopotun Kirill (Canada), Mahmudov Elmkhan (Azerbaijan), Mamedov Farman (Azerbaijan), Melikov Telman (Azerbaijan), Mustafayev Heybetkulu (Azerbaijan), Panahov Geylani (Azerbaijan), Rakhmanov Zafar (Uzbekistan), Salimov Arif (Azerbaijan), Savaş Ekrem (Türkiye), Savchuk Viktor (Ukraine), Shevchuk Igor (Ukraine), Shoikhet David (Israel), Temirgaliev Nurlan (Kazakhstan), Vuorinen Matti (Finland), Zagorodnyuk Andriy (Ukraine), Zajac Jozef (Poland)

Working group:

Head of the group: Mehdi Mammadov

Deputy head of the group: Khalida Melikova

Members:

Aliyev Ramil, Alizada Tahir, Imanova Firangi, Mammadov Ibrahim, Mammadov Vusal, Muradova Shamsiyya, Omarova Aygun, Safarov Zaman, Shikverdiyev Nemat

Technical editor: Tahira Musayeva

ISBN 978-9952-8656-1-5

ALGEBRA, GEOMETRY AND TOPOLOGY**A Note on Perturbed Soft S-Metric Space****Sebuhi Abdullayev^a, Çiğdem Gündüz^b, Sadi Bayramov^a**^a*Baku State University, Baku, Azerbaijan*^b*Kocaeli University, Faculty of Arts and Sciences, Department of
Mathematics, Kocaeli, Türkiye
(e-mail: sebuhi.abdullaye@mail.ru)*

Over the past few years, generalized metric structures have gained considerable interest due to their extensive applications and flexibility in fixed point theory and various branches of mathematics. Specifically, soft S-metric spaces [1] generalize classical metric spaces in a flexible manner. In addition to soft S-metric spaces, perturbed metric spaces [2] have also received increasing attention in the literature. In light of these developments, we define a new generalized distance structure termed a perturbed soft S-metric space, which unifies the essential features of perturbed metric spaces and soft S-metric spaces within a common framework. In this research, we present the basic definitions of the newly introduced space and investigate several of its topological and analytical properties, such as convergence, Cauchy sequences, and completeness. As an additional contribution, we investigate several fixed point theorems in this space and obtain corresponding fixed point results. We support the obtained results with several illustrative examples. Owing to its flexible and versatile nature, this new generalized distance structure has significant potential for applications in generalized metric space theory and related fields and is expected to make a valuable contribution to the existing literature.

References

- [1] C. Gunduz Aras, S. Bayramov, V. Cafarli, *A Study on Soft S-Metric Spaces*, Communications in Mathematics and Applications, 2018, 9(4), 713-723.
- [2] M. Jleli, S. Bessem, *On Banach's Fixed Point Theorem in Perturbed Metric Spaces*, Journal of Applied Analysis and Computation, 2025, 15(2), 993-1001.

An Extension of the Narayana Sequence to Non-Integer Indices

Furkan Arikan, Orhan Dişkaya

Mersin University, Mersin, Türkiye
(e-mail: furkanarikan5858@gmail.com)

This paper presents an extension of the Narayana sequence to non-integer indices by adapting Halsey's construction for the Fibonacci sequence. A Narayana function is introduced, and its fundamental properties are established (1). Based on this construction, Bunder's results on Halsey's Fibonacci function are reformulated and extended to the Narayana function. In conclusion, various identities and algebraic properties of Halsey's Fibonacci function have been correlated with the Narayana sequence. The proposed framework provides a natural extension of the Narayana sequence beyond integer indices and offers a basis for further investigations into the algebraic properties of Narayana-type sequences (2).

References

- [1] E. Halsey, *The fibonacci number F_u where u is not an integer*, The Fibonacci Quarterly, 1965, 3(2), 147-152.
- [2] N. W. Bunder, *On Halsey's fibonacci function*, The Fibonacci Quarterly, 1975, 13(2), 209-210

Computable and Non-Computable Real Numbers

Marat Arslanov

Kazan Federal University
(e-mail: Marat.Arslanov@kpfu.ru)

Real numbers are represented typically by Dedekind cuts, Cauchy sequences of rational numbers and binary or decimal expansions. The effectivization of

these representations leads to equivalent definitions of computable real numbers. The notion of computable numbers was first explored by Alan Turing in his famous paper "On computable numbers, with an application to the "Entscheidungsproblem". Proceedings of the London Mathematical Society, 42(2):230-265, 1936", where also the notion of Turing machine is introduced.

According to Turing, the computable numbers may be described briefly as the real numbers whose binary expressions are calculable by finite means. In other words, a real number $x \in [0; 1]$ is called computable if there is a computable function $f : N \rightarrow \{0, 1\}$ such that

$$x = \sum_{n \in N} f(n) \cdot 2^{-(n+1)} = 0.f(0)f(1)f(2)\dots\dots$$

In my talk, I'm going to speak about the possibilities of computer representation of noncomputable real numbers: real numbers that are approximable in limit by computable functions, computable functions, with a certain probability calculating uncomputable real numbers, and so on.

Some Remarks on Uniform Algebras Valued Composition Operator

Ali Babaev^a, Vafa Mammadova^a, Zumrud Panahova^b, Eminaga
Mamedov^a

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan State Pedagogical University, Baku, Azerbaijan*

(e-mail: a.babayev49@gmail.com, vafa.mamedova@yahoo.com,
zumrudpanahova@gmail.com, eminm62@gmail.com)

1. Introduction. Theory of uniform algebras is a part of the theory of commutative Banach algebras, the latter created by I. Gelfand in the 1940s. This theory has been rapidly developing since then. Shilov's articles (1950s) should be mentioned as one of the first works in this direction. R. Arens, V. Zinger, A. Gleason, E. Bishop, J. Verner, etc. also largely contributed to the development of this theory.

Banach algebras date back to the 1930s when von Neumann introduced a special type of algebras. In 1939-1941, Gelfand created the theory of commutative Banach algebras. In 1943, Gelfand and Naimark introduced the concept

of C^* -algebras based on the special role of involution. In particular, the following theorems have been proved:

1. (Commutative case, Gelfand-Naimark). Every commutative C^* -algebra can be identified with the algebra $C_0(\Omega)$ of continuous functions defined on some locally compact space with the accuracy of isometric $*$ -isomorphism of algebras.

2. (Non-commutative case, Gelfand-Naimark). Any C^* -algebra can be identified with some operator algebra with the accuracy of isometric $*$ -isomorphism of algebras.

2. Useful informations. Let us introduce some needful information which will be used in this report.

A topological space X is said to be **completely regular** if for arbitrary x in X and any closed subset A of X not containing x there exist a continuous function $f : X \rightarrow [0, 1]$, such that $f(x) = 1$ and $f(y) = 0$ for all y in A . A **Banach function algebra on a compact Hausdorff space** X is a Banach algebra A consisting of continuous functions on X such that A separates points of X and contains the constant functions. A **uniform algebra** on a compact Hausdorff space X is a uniformly closed subalgebra of $C(X)$, which contains the constants and separates the points of X . Two points $S_0(A(X))$ of a topological space $S_0(A(X))$ are called **compactly connected** if there exists a connected compact set $S(A(X))$, such that it contains both of these points. A subset K of X is called a **peak set** for A if there is an $f \in A$ such that $f \equiv 1$ on K and $|f(x)| < 1$ for $x \notin K$. If $K = \{x_0\}$, then x_0 is called a peak point. Let $A(X)$ be a uniformly closed subspace of $C(X)$ (in particular, a uniform algebra). A closed subset $E \subset X$ is called a **peak set with respect to $A(X)$** , if there exists a sequence $\{f_n\} \subset A(X)$ such that $\|f_n\| = f_n(x) = 1$ for all n , for all $x \in E$ and moreover, outside any neighborhood of the set E the sequence $\{f_n\}$ tends to 0 uniformly. A peak set consisting of only one point is called a **peak point with respect to $A(X)$** . Denote the set of all peak sets with respect to $A(X)$ by $S(A(X))$, and the set of all peak points with respect to $A(X)$ by $S_0(A(X))$. We will assume that the set $S_0(A(X))$ is dense in $S(A(X))$ and the number of the compactly connected components of $S(A(X))$ is finite. Let $A(X)$ be a uniformly closed subspace of $C(X)$ (in particular, a uniform algebra). A mapping $\varphi : X \rightarrow X$ is called a **compositor on $A(X)$** , if $f \circ \varphi \in A(X)$, whenever $f \in A(X)$. A function $u \in C(X)$ is called a **multiplicator with respect to $A(X)$** if $u \cdot f \in A(X)$ for all functions

$f \in A(X)$. We denote the set of all compositors on $A(X)$ by $C_{A(X)}$, and the set of all multipliers with respect to $A(X)$ by $M_{A(X)}$.

Convention. We will only consider the following case: let $C(X)$ be a Banach algebra of continuous functions, $A = A(X) \subset C(X)$ be a subalgebra, and $C(X, A)$ be the algebra of A -valued continuous functions on X with the norm $\|h\|_{C(X,A)} = \sup_{x \in X} \|h(x)\|_{C(X)}$. Also, let $\varphi : X \rightarrow X$ be a self-mapping of X , and $u \in C(X, A)$ be a given function. Consider the weighted composition operator $T : C(X, A) \rightarrow C(X, A)$ defined as follows:

$$(T \cdot f)(x) = u(f \circ \varphi)(x), f \in C(X, A), u \in M_{A(X)}, \varphi \in C_{A(X)}. \quad (1)$$

In the sequel, by $s(u)$ we will denote $s(u) = \{x \in X : \exists u^{-1}(x)\}$.

We assume that

$$\forall x \in X \Rightarrow u(x) = 0 \vee \exists u^{-1}(x).$$

We will also assume that $\varphi|_{s(u)}$ is a continuous function on $s(u)$. From the classical facts it follows that $s(u)$ is an open set of X . And if for $\forall x \in X \Rightarrow u(x) = u(x_0)$, then

$\forall g(\cdot) \in C(X, A(X)) \Rightarrow u(x)g(x) = u(x_0)g(x) \in C(X, A(X))$
i.e. in $A(X)$ multiplication by “constant function” is possible.

3. Main results. In [1] the following statements are obtained.

Theorem 1. *Let the operator T defined by (2.1) be a compact operator. Then for any compactly connected component $K \subset s(u)$ and for any peak set E with respect to $A(X)$, we have either $\varphi(K) \subseteq E$, or $\varphi(K) \cap E = \emptyset$.*

Our main aim is the following theorems.

Theorem 2. *Let X be some compact set, $C(X)$ is a uniform algebra. Then for compactness of $C(X)$ -valued weighted composition operator (for endomorphism) $T : C(X, C(X)) \rightarrow C(X, C(X)) : T = u(f \circ \varphi)$ it is necessary and sufficient: for arbitrary compact subset $K \subset s(u)$ the set $\varphi(K)$ will be a finite set.*

Theorem 3. *Let X be a local connected compact set. $C(X, A)$ is a topological space of all continuous A -valued functions with compact-open topology, where $A = A(X)$ is a subalgebra of $C(X)$, and $T = u(f \circ \varphi)$ is a A -valued weighted composition operator. And also assume that for arbitrary compact set $P \subset X$, which non intersected with $S_0(A(X))$ the restriction of $A(X)$ on the $C(P)$, is a compact operator. Then for compactness of the operator T it*

is necessary and sufficient: for each connected compact component $K \subset s(u)$ either $\varphi(K)$ is a finite set, or $\varphi(K) \subseteq X \setminus S_0(A)$.

Theorem 4. Let X be local connected set. $A(X) \subset C(X)$ is a subspace from Theorem 3.3 and the peak point set $S_0(A(X)) \subset X$ coincides with topological boundaries of ∂X (i.e. $S_0(A(X)) = \partial X$), and let $G = X \setminus S_0(A(X))$ is nonempty. Then for compactness of the weighted composition operator T it is necessary and sufficient: for arbitrary compact connected subset $K \subset s(u)$ will be satisfied the following relation: $\varphi(K) \subset X \setminus S_0(A(X))$.

References

- [1] Shahbazov A., Panahova Z., Mammadova V. Transactions of Pedagogical University, 72(3), (2024), 10-18.

The Concept of Quantity and Equality in the Works of Nasir Al-Din Al-Tusi

Rena Babayeva^a, Sabir Mirzoev^b, Eminaga Mamedov^a

^aInstitute of Mathematics, Baku, Azerbaijan

^bBaku State University, Baku, Azerbaijan

(e-mail: renababayeva1427@gmail.com, eminm62@gmail.com)

In this report we briefly study al-Tusi's views concerning quantity and equality. A precise understanding of some of his views is impossible without studying his philosophical and logical heritage. In this respect, in our opinion, the following works may be regarded as useful sources:

1. "Tajrid al-i'tiqad" is considered his main philosophical work. The book consists of six sections. The first two sections are devoted to questions of natural philosophy and contain al-Tusi's views on the Universal, matter, the construction of matter, substances, accidents, questions of motion, time, space and many others. The work was commented on by many scholars. Among these commentaries the commentary of Hilli (al-Hasan ibn Yusuf al-Mutahhar al-Hilli, d. 1322), a student of al-Tusi, is widely known under the title "Kashf al-murad fi sharh tajrid al-i'tiqad" [1].

2. “Sharh al-Isharat wa al-Tanbihat” [2] is a commentary by al-Tusi on Ibn Sina’s treatise “The Book of Directives and Remarks”.

3. “Asas al-Iqtibas” is considered the main work of al-Tusi on logic.

4. “Tajrid al-Mantiq” (“The Abstraction of Logic,” in Arabic) [3] is a summary of the course of logic. There are Hilli’s commentaries on this work under the title “Al-Jawhar al-Nazid fi Sharh Mantiq al-Tajrid” (“Systematic Foundations of Explanations of the Logic of Abstraction”) [4].

In parallel with this, we compare al-Tusi’s main theses with Hegel’s theses based on [5] and al-Farabi’s theses based on [6].

We would like to draw attention to one fundamental difference in the views of al-Farabi, al-Tusi and Hegel. It is revealed in questions of equality.

In the chapters [3, 4] the following is found about definitions: “Essential accidents can be determined only by recalling what they manifest themselves in” [4, p. 347]. In Hilli’s commentaries there is: “as, for example, when we wish to define equality, in its definition we take what it manifests itself in — that is, quantity. Equality is quantitative compatibility (commensurability)”. Thus, for Tusi equality is a quantitative relation. Consequently, equality is a quantitative characteristic, not a qualitative characteristic.

As is known, Euclid’s axioms of order and equality are the following [8, Book I]: For convenience we shall write $A > B$ ($A = B$) if the measure A is greater than (equal to) the measure B , and union (disjunction) and equivalence are expressed respectively by “ $\vee$ ” and “ $\Leftrightarrow$ ”.

1. $A = B, C = B \Rightarrow A = C$. 2. $A = B \vee C \Rightarrow A + C = B + C$ (in Tusi this is in one proposition [9]). 3. $A \neq B \vee C \Rightarrow A \pm C \neq B \pm C$. 4. $A = B \Rightarrow 2A = 2B$. 5. $A = B \Rightarrow A/2 = B/2$. 6. Things coinciding with one another are equal to one another. 7. The whole is greater than the part, i.e. $A \supset B \Rightarrow A > B$.

In Tusi [9] there is an additional proposition 4’: $A + C = B + C \vee A - C = B - C \Rightarrow A = B$. Among these axioms there is no discrete axiom of reflexivity, i.e. $A = A$.

Hegel has the following thesis: “Quantity, taken in its immediate relation to itself, or, in other words, its immediate attraction determined by equality with itself, is a continuous quantity; but taken in another way, as containing in itself the determined one, it is a discrete quantity”. Yet the first quantity is also discrete, for it is only the continuity of many; and the second is also continuous and not continuous, for it is like the identity of many ones, like a unit. Note 1)

It does not follow, therefore, that continuous and discrete quantities should be considered as species, one of which does not possess the definition of the other; in fact, they differ from one another only by the fact that the same whole is once posited under one of its determinations, and another time under another. ([5], p. 246). The phrases "... equality with itself ..." and "the One as identical to many ones, as a unit" define the reflexivity of the concept of equality. That is, the concept of quantity is a class factor of mutually comparable quantities. Therefore every quantity is equal to itself.

In the thesis cited above, al-Farabi's academic continuous quantities, as concepts, act as expressions of the material. They are a generalized image of the real in the imagination, i.e. an ideal model of single things in consciousness. In this case the objects themselves are not identified. Therefore the property of reflexivity is not taken as an axiom for the relation of equality.

As an example, let us turn to proposition 5 from [8].

Proposition 5. In isosceles triangles, the angles at the base are equal to one another; and, when the equal straight lines are produced, the angles under the base will also be equal to one another.

In other words, if ABC is an isosceles triangle with base BC , then angles B and C are equal. Before this, proposition 4 is proved: if two triangles have two sides respectively equal to two sides and the angle contained by the equal straight lines equal to the corresponding angle, then the bases will be equal, one triangle will be equal to the other, and the remaining angles subtended by the equal sides will be equal respectively. (Criterion of equality of two triangles by two sides and the angle between them.).

One might expect the application of Proposition 4 to prove Proposition 5 among al-Tusi's comments in [9]. Starting from the fact that triangles ABC and ACB are equal to one another, and therefore angles B and C must be equal to one another. That is, one would only have to refer to the reflexivity of the relation of equality of figures.

Thus, the relation of equality in al-Tusi also does not possess the property of reflexivity.

References

- [1] Hilli, Kashf al-Murad fi Sharh Tajrid al-I'tiqad. Tehran, 2004–2005, 592p.

- [2] N. Tusi, Sharh al-Isharat. Cairo, 1960, 567 p.
- [3] N. Tusi. Tajrid al-Mantiq. Beirut, 1988, 79 p.
- [4] Hilli, Al-Jawhar al-Nazid fi Sharh Mantiq al-Tajrid Qom, Iran, 2006, 480 p.
- [5] Hegel, Encyclopedia of the Philosophical Sciences. Vol. 1. Science of Logic. Moscow: Mysl, 1974.
- [6] Al-Farabi, Ihsa al-Ulum Institute of Manuscripts of the National Academy of Sciences of the Azerbaijan Republic. p. 92-115, B-94.
- [7] N. Tusi, Treatise on the Complete Quadrilateral (Shakl al-qita). Translation edited by G. D. Mamedbeyli and B. A. Rozenfeld. Baku, 1952.
- [8] N. Tusi. Tahrir usul Uqlidis fi Ilm al-Handasa Tehran version, 1881, 210 p.

Some Relations Between Trigonometric Functions and Algebraic Polynomials

Elmaga Gasymov, Arifa Kerimova

Baku State University, Baku, Azerbaijan

(e-mail: elmagagasymov@gmail.com, kerimovaarifa79@gmail.com)

In the paper using trigonometric functions some high-order algebraic polynomial is constructed. All the roots of this polynomial are found and it is factorized into elementary factors. Using this factorization we obtain a suitable formula for finding some trigonometric products.

Using mathematical induction method we prove [2] the following.

Theorem 1. *For any natural number n the following identities hold:*

$$\begin{aligned}\sin(2n+1)\varphi &= \sin\varphi \sum_{k=0}^n a_k \sin^{2k}\varphi, \\ \cos(2n+1)\varphi &= \cos\varphi \sum_{k=0}^n b_k \cos^{2k}\varphi,\end{aligned}\tag{1}$$

where φ is arbitrary angle, a_k, b_k are some numbers and $a_n = b_n = (-1)^n 2^{2n}$; $a_0 = 2n + 1$; $b_0 = 1$

Let

$$E_{2n+1}(t) = t \sum_{k=0}^n a_k t^{2k}, \quad (2)$$

where a_k is from (1) ($E_{2n+1}(t)$ is Elmag polynomial)

From the method of determining the polynomial $E_{2n+1}(t)$ we obtain the following identity

$$E_{2n+1}(\sin \alpha) = \sin(2n + 1)\alpha, \quad (3)$$

where α is arbitrary angle.

Let's consider the equation

$$\sin(2n + 1)\alpha = 0. \quad (4)$$

From (4) we get

$$(2n + 1)\alpha = k\pi, k = 0, \pm 1, \pm 2, \dots$$

Therefore, numbers t_k defined by formulas

$$\begin{aligned} t_0 = 0, t_k &= \sin \frac{k\pi}{2n+1}, k = 1, 2, \dots, n; \\ t_{n+s} &= -\sin \frac{s\pi}{2n+1}, s = 1, 2, \dots, n; \end{aligned} \quad (5)$$

are the roots of the equation

$$E_{2n+1}(t) = 0. \quad (6)$$

From the algebra course it is known [1] that if $t_0, t_1, \dots, t_{2n}$ are the roots of algebraic equation (6) then for any t the identity

$$E_{2n+1}(t) = a_n(t - t_0)(t - t_1)\dots(t - t_{2n}). \quad (7)$$

holds.

Taking (5) into account in (7) we get the identity

$$E_{2n+1}(t) = a_n t \prod_{k=1}^n \left(t - \sin \frac{k\pi}{2n+1} \right) \left(t + \sin \frac{k\pi}{2n+1} \right). \quad (8)$$

Thus we prove the following

Theorem 2. For polynomial $E_{2n+1}(t)$ defined by formula (2) the identity (8) holds.

Assuming $t = \sin \alpha$ in (8) and using the identity (3) we get

$$\sin(2n+1)\alpha = a_n \sin \alpha \prod_{k=1}^n \left(\sin \alpha - \sin \frac{k\pi}{2n+1} \right) \left(\sin \alpha + \sin \frac{k\pi}{2n+1} \right).$$

From here

$$\sin \alpha \prod_{k=1}^n \left(\sin^2 \alpha - \sin^2 \frac{k\pi}{2n+1} \right) = \frac{(-1)^n}{2^{2n}} \sin(2n+1)\alpha. \quad (9)$$

Thus the following theorem is proved.

Theorem 3. For the product

$$\sin \alpha \prod_{k=1}^n \left(\sin^2 \alpha - \sin^2 \frac{k\pi}{2n+1} \right), \quad (10)$$

the identity (9) holds, where n is arbitrary natural number, α is arbitrary angle.

Formula (9) can be used for finding product (10) and also for finding values of trigonometric functions

For example, at $n = 2$ the identity (9) takes the form:

$$\sin \alpha (\sin^2 \alpha - \sin^2 36^\circ) (\sin^2 \alpha - \sin^2 72^\circ) = \frac{1}{16} \sin 5\alpha. \quad (11)$$

Using formula

$$\sin 5\alpha = \sin \alpha (16 \sin^4 \alpha - 20 \sin^2 \alpha + 5)$$

from (11) we get

$$5x - 4x^2 = \frac{5}{4}, \quad (12)$$

where $x = \sin^2 36^\circ$.

Solving quadratic equation (12) we find

$$\sin 36^\circ = \sqrt{\frac{5 - \sqrt{5}}{8}}.$$

References

- [1] A.G. Kurosh, *Course of higher algebra*, M. Nauka, 1975, 431 p.
- [2] E.A. Gasymov, *Scientific foundations of the course of elementary mathematics*, Baku Elm, 2016, 497 p.

Curvature Properties of the Fiberwise Berger type Deformed Sasaki Metric on the Coframe Bundle

Habil Fattayev

Baku State University, Baku, Azerbaijan
(e-mail: h-fattayev@mail.ru)

Great interest in the Sasaki metric [1] and in the differential-geometric structures associated with it led to the idea of constructing new modifications of the Sasaki metric in fiber bundles (see, for example, [2], [3]). The main goal of this talk is to define the fiberwise Berger type deformed Sasaki metric on linear coframe bundle of a Riemannian manifold and study its curvature properties.

Let (M, φ, g) be an almost anti-para-Hermitian manifold and $F^*(M)$ be its coframe bundle [4]. A fiberwise Berger type deformed Sasaki metric is a Riemannian metric ^{BS}g on the linear coframe bundle $F^*(M)$ such that

$$\begin{aligned} ^{BS}g(^HX, ^HY) &= ^V(g(X, Y)) = g(X, Y), \\ ^{BS}g(^{V_\alpha}\omega, ^HY) &= 0, \\ ^{BS}g(^{V_\alpha}\omega, ^{V_\beta}\theta) &= 0, \quad \alpha \neq \beta, \\ ^{BS}g(^{V_\alpha}\omega, ^{V_\alpha}\theta) &= g^{-1}(\omega, \theta) + \delta^2 g^{-1}(\omega, X^\alpha \varphi) g^{-1}(\theta, X^\alpha \varphi) \end{aligned}$$

for all $X, Y \in \mathfrak{F}_0^1(M)$ and $\omega, \theta \in \mathfrak{F}_1^0(M)$, where δ is some constant [5].

The basic properties of the Levi-Civita connection $^{BS}\nabla$ of the metric ^{BS}g are investigated, and the components of the Riemannian curvature tensor of the linear coframe bundle $F^*(M)$ with the metric ^{BS}g are calculated. In particular, the following theorem is proved.

Theorem. Let (M, φ, g) be an anit-para-Kahler manifold and ${}^{BS}\nabla$ be the Levi-Civita connection of the linear coframe bundle $F^*(M)$ equipped with the metric ${}^{BS}g$. The particular values of components for different indices are then found to be

$$\begin{aligned} {}^{BS}\Gamma_{ij}^k &= \Gamma_{ij}^k, & {}^{BS}\Gamma_{ij}^{k_\gamma} &= \frac{1}{2}X_m^\gamma R_{ijk}^m, \\ {}^{BS}\Gamma_{ij\beta}^k &= \frac{1}{2}X_m^\beta R_{\cdot i \cdot}^{k \cdot jm}, & {}^{BS}\Gamma_{ij\beta}^{k_\gamma} &= -\delta_\gamma^\beta \Gamma_{ik}^j, \\ {}^{BS}\Gamma_{i\alpha j}^k &= \frac{1}{2}X_m^\alpha R_{\cdot j \cdot}^{k \cdot im}, & {}^{BS}\Gamma_{i\alpha j}^{k_\gamma} &= 0, \\ {}^{BS}\Gamma_{i\alpha j\beta}^k &= 0, & {}^{BS}\Gamma_{i\alpha j\beta}^{k_\gamma} &= 0 \text{ for } \alpha \neq \beta, \quad {}^{BS}\Gamma_{i\alpha j\alpha}^k = 0, \\ {}^{BS}\Gamma_{i\alpha j\alpha}^{k_\gamma} &= \frac{\delta^2}{1+\delta^2 g^{-1}(X^\alpha, X^\alpha)} g^{sj} \varphi_s^i X_m^\gamma \varphi_k^m. \end{aligned}$$

References

- [1] S. Sasaki, *On the differential geometry of tangent bundle of Riemannian manifolds*, Tohoku Math. J. 1958, 10, 338-358.
- [2] A. Gezer and M. Altunbas, *Notes on the rescaled Sasaki type metric on the cotangent bundle*, Acta Math. Sci. Ser. B Engl. Ed, 2014, 34, 162-174.
- [3] A.A. Salimov and A. Filiz, *Some properties of Sasakian metrics in cotangent bundles*, Mediterr. J. Math., 2011, 8, 243-255.
- [4] H.D. Fattayev and A.A. Salimov, *Diagonal lifts of metrics to coframe bundle*, Proc. of the IMM, NAS of Azerbaijan, 2018, 44(2), 328-337.
- [5] A. Zagane, *Berger Type Deformed Sasaki Metric and Harmonicity on the Cotangent Bundle*, International Electronic Journal of Geometry, 2021, 14(1), 183-195.

A k-Valued Discrete-Probability Logic of Event Formulas Without Combinatorial Explosion

Vagif Jabbarzade

Azerbaijan State Pedagogical University, Baku, Azerbaijan
(e-mail: jabbarzade@gmail.com)

This work is a continuation of the work in [2-3].

Consider an experiment S [1] having various outcomes, and suppose that the complete list of possible outcomes and their probabilities is known in advance. We consider the case in which the set of distinct outcomes is finite. The distinct outcomes are called elementary events and are denoted by ω_i ($i = 1, 2, \dots, n$). The set of elementary events, denoted by the capital Greek letter $\Omega = \{\omega_1, \dots, \omega_n\}$, is called the probability space. It is assumed that each outcome ω_i ($i = 1, 2, \dots, n$) can take three values:

- if $\varphi(\omega_i) = 0$, the elementary event ω_i did not occur;
- if $\varphi(\omega_i) = 1$, the elementary event ω_i occurred with probability $P(\omega_{i1}) \geq 0$;
- if $\varphi(\omega_i) = 2$, the elementary event ω_i occurred with probability $P(\omega_{i2}) \geq 0$.

$$P(\omega_{i1}) + P(\omega_{i2}) = P(\omega_i), \quad P(\omega_1) + \dots + P(\omega_n) = 1.$$

Lukasiewicz logical operations, adapted from [5] by replacing $1/2$ with 2 for convenience in further data processing.

Remark 1. In the realization of experiment S , exactly one of the elementary events ω_i ($i = 1, 2, \dots, n$) can occur. That is, if $\varphi(\omega_i) = 1$ or 2 for some i , then $\varphi(\omega_j) = 0$ for every $j \neq i$.

Definition 1.

1. Any elementary event $\omega_i \in \Omega$ ($i = 1, 2, \dots, n$) is an event formula.
2. If A is an event formula, then $\neg A$ is an event formula.
3. If A and B are event formulas, then $(A \vee B)$ and $(A \wedge B)$ are event formulas.

4. An event formula can be obtained only by applying rules 1, 2, and 3.

The set of all event formulas of the three-valued Lukasiewicz logic is denoted by $SF3E$.

Let B be a formula of events of rank m , composed of the non-fictitious elementary events $\omega_{k_1}, \dots, \omega_{k_m}$.

Algorithm for computing the probability of an event formula B (Lukasiewicz). [*]

Definition 2-5.

$$P(B^{(1)}) = \sum_{i=0}^m P(\omega_{k_{2i+1}}) \mid (\text{in the } (2i+1)\text{-st row, } \varphi(B) = 1).$$

$$SUEE(B^{(1)}) = \bigcup_{i=0}^m \{\omega_{k_{2i+1}}\} \mid (\text{in the } (2i+1)\text{-st row, } \varphi(B) = 1).$$

$$P(B^{(2)}) = \sum_{i=1}^{m+1} P(\omega_{k_{2i}}) \mid (\text{in the } (2i)\text{-th row, } \varphi(B) = 2).$$

$$SUEE(B^{(2)}) = \bigcup_{i=1}^{m+1} \{\omega_{k_{2i}}\} \mid (\text{in the } (2i)\text{-th row, } \varphi(B) = 2).$$

The probability of the formula of events B is $P(B) = P(B^{(1)}) + P(B^{(2)})$.

If every value in the column $\varphi(B)$ of the table equals 0, then $P(B^{(1)}) = P(B^{(2)}) = P(B) = 0$.

Property 1. For any event formula B , $0 \leq P(B) \leq 1$. The validity of this inequality follows from the algorithm for computing $P(B)$.

Property 2. For any event formulas A and B , and for each $r \in \{1, 2\}$,

$$P(A^{(r)} \vee B^{(r)}) = P(A^{(r)}) + P(B^{(r)}) - P(A^{(r)} \wedge B^{(r)}).$$

The proof of these equalities follows from the proof of equality (5).

Property 3. Hence, for each $r \in \{1, 2\}$, if $SUEE(A^{(r)} \cap B^{(r)}) = \emptyset$, then

$$P(A^{(r)} \vee B^{(r)}) = P(A^{(r)}) + P(B^{(r)}).$$

Proofs of Properties 1–3 are given in [2].

Definition 6. If $SUEE(Q^{(r)}) \subseteq SUEE(B^{(r)})$, then we say that the event formula $B^{(r)}$ probably logically follows from the event formula $Q^{(r)}$ with probability $P(Q^{(r)})$, where $r \in \{1, 2\}$.

Definition 7. If $SUEE(Q_1^{(r)} \wedge \dots \wedge Q_m^{(r)}) \subseteq SUEE(B^{(r)})$, then we say that the event formula B probably logically follows from the event formulas $Q_1^{(r)}, \dots, Q_m^{(r)}$ with probability $P(Q_1^{(r)} \wedge \dots \wedge Q_m^{(r)})$, where $r \in \{1, 2\}$.

Definition 8. $L'_3 = (\{0, 1, 2\}, \Omega, SF3E, P)$ is called a three-valued probabilistic logical space of Jan Lukasiewicz, where $SF3E$ is the set of all event formulas of the three-valued Lukasiewicz logic.

Remark 2. A table of formulas with m variables in the three-valued Lukasiewicz logic consists of 3^m rows, whereas a table of event formulas in the three-valued probabilistic logic of event formulas of Jan Lukasiewicz with m variables consists of $2(m + 1)$ rows.

Conclusion. In this paper, a three-valued logic of events is constructed that does not lead to a combinatorial explosion. A k -valued logic of events can be constructed in a similar manner for $k = 2$ and $k \geq 4$. In k -valued logic, given m events, the event table contains $(k - 1)(m + k - 2)$ rows, avoiding combinatorial explosion, instead of k^m rows.

References

- [1] Grinstead, C. M., and Snell, J. L. *Introduction to Probability*. 2nd ed., American Mathematical Society, 2003.
- [2] Jabbarzade, V. *Probabilistic Logic of k -Valued Formulaic Events Without Combinatorial Explosion*. Published June 13, 2026.
- [3] Jabbarzade, V. *Construction of a Factor-Probabilistic Propositional Calculus Based on the 8-Valued Logic*. Proceedings of the International Conference “Modern Problems of Mathematics and Mechanics,” Baku, Azerbaijan, September 03–06, 2025, pp. 31–34.
- [4] Lukasiewicz, J. *Selected Works*. North-Holland Publishing Company, Amsterdam, 1970.

Topologies on Locally Finite Graphs

Ljubiša Kočinac

University of Niš, 18000 Niš, Serbia
(e-mail: lkocinac@gmail.com)

We consider two topologies on the vertex sets $V(G)$ of simple locally finite graphs. The first topology, called $\mathcal{P}$ -topology, is given on an undirected graph G , and the second topology, called $\mathcal{F}$ -topology, is defined on a directed graph G . Several properties of these topologies are established. An application of the $\mathcal{F}$ -topology to the cardiovascular system is presented.

References

- [1] H. Alzubaidi, D. Djurčić, Lj. D. R. Kočinac, H. A. Othman, *$\mathcal{P}$ -topological spaces in simple graphs*, Journal of Mathematics and Computer Science, 2026, 40(3), 330–340.

On Uniform Structures in the Superextension

Farkhod Mukhamadiev

National University of Uzbekistan named after Mirzo Ulugbek, str. University 4, 100174 Tashkent, Uzbekistan
(e-mail: fmukhamadiev.fm@gmail.com)

In this work, we consider uniform structures in the superextension of topological space X .

Let X be a non-empty set, and α and β are covers of the set X . It is said that α is inscribed in the cover of β , if for any $A \in \alpha$ there exists $B \in \beta$, such that $A \subset B$, and write $\alpha \succ \beta$.

Let α be the cover of the set X and $Y \subset X$. The set $\alpha(Y) = \bigcup \{A \in \alpha : A \cap Y \neq \emptyset\}$ is called the star set with respect to the cover α . If $Y = \{x\}$, then we will write $\alpha(x)$ instead of $\alpha(\{x\})$.

Next, put $\alpha^\Delta = \{\alpha(x) : x \in X\}$, $\alpha^* = \{\alpha(A) : A \in \alpha\}$. They say that a cover α is star inscribed in β , if $\alpha^\Delta \succ \beta$. If $\alpha^* \succ \beta$, then they say that α is strongly star inscribed in β (see [1]).

Definition 1 [1]. Let X be a nonempty set. A family $\mathcal{U}$ of coverings of a set X is called uniformity on X if the following conditions are satisfied:

- (1) If $\alpha \in \mathcal{U}$ and α is inscribed in some cover β of the set X , then $\beta \in \mathcal{U}$.
- (2) For any $\alpha_1 \in \mathcal{U}$, $\alpha_2 \in \mathcal{U}$ there exists $\alpha \in \mathcal{U}$, which is inscribed in α_1 and α_2 .
- (3) For any $\alpha \in \mathcal{U}$, there exists $\beta \in \mathcal{U}$ strongly star inscribed in α .
- (4) For any x, y of a pair of different points of X , there exists $\alpha \in \mathcal{U}$ such that no element of α contains both x and y .

The pair $(X, \mathcal{U})$ is called a uniform space.

For any uniformity of $\mathcal{U}$ on X , the family

$\tau_{\mathcal{U}} = \{O \subset X : \text{for each } x \in O \text{ exists } \alpha \in \mathcal{U} \text{ such that } \alpha(x) \subset O\}$ is a topology on X and the topological space $(X, \tau_{\mathcal{U}})$ is a T_1 -space. The topology of $\tau_{\mathcal{U}}$ is called the topology generated or induced by the uniformity of $\mathcal{U}$.

A system $\xi = \{F_\alpha : \alpha \in A\}$ of closed subsets of a space X is called linked, if any two elements of ξ intersect.

Any linked system can be upgraded to a maximal linked system (MLS). But such upgrade, as a rule, is not one valued. A linked system of space is MLS, if and only if it possesses the following completeness property [2]: "If a closed set $A \subset X$ intersects with every element of ξ , then $A \in \xi$ ".

We denote λX as the set of all MLS of the space X .

For the close set $A \subset X$ we consider $A^+ = \{\xi \in \lambda X : A \in \xi\}$.

For the open set $U \subset X$ we consider

$$O(U) = \{\xi \in \lambda X : \text{there exists } F \in \xi \text{ such that } F \subset U\}.$$

The family of sets of the form $O(U)$ covers the set λX ($O(X) = \lambda X$). So, it forms an open prebase of the topology on λX . The set λX , equipped with this topology, is called as the superextension of the space X [2].

Let X be topological space and λX be its superextension. MLS $\xi \in \lambda X$ is called compact, if it contains at least one compact element, and is denoted by CMLS.

The space $\lambda_c X = \{\xi \in \lambda X : \xi \text{ is CMLS}\}$ we call as compact super kernel (or compact superextension) of the topological space X .

It is clear that $\lambda_c X \subset \lambda X$. We see that $\lambda^* X \subseteq \lambda_c X \subseteq \lambda X$ for topological T_1 -space X .

If the space X is compact, then we have the equality $\lambda_c X = \lambda X$. If the space X is discrete, then we have another equality $\lambda^* X = \lambda_c X$.

The basement of the CMLS ξ in X is the family $F(\xi) = \{F \in \xi : F \text{ is a compact}\}$ [3].

Let $(X, \mathcal{U})$ be a uniform space and λX is the superextension of the space $(X, \tau_{\mathcal{U}})$.

For each $\alpha \in \mathcal{U}$, put $P(\alpha) = \{\langle \alpha' \rangle : \alpha' \subseteq \alpha\}$, where

$$\langle \alpha' \rangle = \{\xi \in \lambda X : \forall A \in \alpha', \exists F \in \xi : F \subset A\}.$$

Proposition. If $\mathcal{B}$ is the base of a uniform space $(X, \mathcal{U})$, then $P(\mathcal{B}) = \{P(\alpha) : \alpha \in \mathcal{B}\}$ forms a base of some uniformity $\lambda \mathcal{U}$ on λX .

Remark. Let $\lambda_c X$ be the compact super kernel of the uniform space $(X, \mathcal{U})$. For each $\alpha \in \mathcal{U}$, put $K(\alpha) = \{\langle \alpha' \rangle : \alpha' \subseteq \alpha \text{ and } \alpha' - \text{finite}\}$. Note that $K(\alpha)$ is the cover of the set $\lambda_c X$.

A finite sequence $\{A_1, A_2, \dots, A_n\}$ of subsets of a set X is called linked if $A_i \cap A_{i+1} \neq \emptyset$ of each $i = 1, 2, \dots, n-1$ [1].

Corollary. If the uniform space $(X, \mathcal{U})$ is metrizable, then its superextension $(\lambda X, \lambda \mathcal{U})$ is also metrizable.

Theorem 1. A uniform space $(X, \mathcal{U})$ is uniformly paracompact if and only if a uniform space $(\lambda_c X, \lambda_c \mathcal{U})$ is uniformly paracompact.

Theorem 2. A uniform space $(X, \mathcal{U})$ is uniformly P -paracompact if and only if a uniform space $(\lambda_c X, \lambda_c \mathcal{U})$ is P -uniformly paracompact.

Theorem 3. A uniform space $(X, \mathcal{U})$ is uniformly B -paracompact if and only if a uniform space $(\lambda_c X, \lambda_c \mathcal{U})$ is B -uniformly paracompact.

Theorem 4. A uniform space $(X, \mathcal{U})$ is uniformly A -paracompact if and only if a uniform space $(\lambda_c X, \lambda_c \mathcal{U})$ is A -uniformly paracompact.

Theorem 5. A uniform space $(X, \mathcal{U})$ is uniformly I -paracompact if and only if a uniform space $(\lambda_c X, \lambda_c \mathcal{U})$ is I -uniformly paracompact.

Theorem 6. A uniform space $(X, \mathcal{U})$ is uniformly R -paracompact if and only if a uniform space $(\lambda_c X, \lambda_c \mathcal{U})$ is R -uniformly paracompact.

References

- [1] A.A. Borubaev, *Uniform topology*, Bishkek. Ilim, (2013).

- [2] V.V. Fedorchuk, V.V. Filippov, *General topology. Basic constructions*, Moscow, Fizmatlit, (2006).
- [3] F.G. Mukhamadiev, *Local τ -density of the sum and the superextension of topological spaces*, Differential equations, geometry, and topology, Itogi Nauki i Tekhniki. Sovrem. Mat. Pril. Temat. Obz., 2021, 201, 103-106.

Topological Games in Bitopological Spaces: The Banach-Mazur Game

Selma Özçağ, Haydar Saraf

Hacettepe University, Department of Mathematics, Beytepe Ankara, Türkiye
(e-mail: sozcag@hacettepe.edu.tr, haidardhjafar@gmail.com)

The Banach-Mazur game is one of the most fundamental topological games and provides a game-theoretic characterization of Baire spaces. The Banach-Mazur game is an infinite two-player game in which the players alternately choose nested non-empty open sets. Classical examples, such as the space of rational numbers and complete metric spaces, show that how the existence of winning strategies depend on the underlying topology.

We define the Banach-Mazur game on a bitopological space and discuss the relationship between the existence of winning strategies and the Baire property with respect to the two topologies.

Acknowledgment. This work is supported by the Scientific Research Projects Coordination Unit of Hacettepe University (Project No: FHD-2025-21908).

References

- [1] R. Telgársky, *Topological Games: On the 50th Anniversary of the Banach-Mazur Game*, Rocky Mountain J. Math. 1987, (17) 227–276.
- [2] L.F. Aurichi, R.R. Dias, *A minicourse on topological games*, Topology and its Appl. 2019, (258) 305–335.

Homology Theory of Soft Sets in the Category of Shostak Topological Spaces

Kamala Veliyeva^a, Nuride Museyibova^b, Sadi Bayramov^c

^aAzerbaijan State Oil and Industry University, Baku, Azerbaijan

^bAzerbaijan State Pedagogical University, Baku, Azerbaijan

^cBaku State University, Baku, Azerbaijan

(e-mail: kamala.valiyeva@asoiu.edu.az, nuride.museyibova@adpu.edu.az, baysadi@gmail.com)

Let $SS(X_E)$ be a family of soft sets over X . A mapping $\tau : SS(X_E) \rightarrow [0, 1]$ is called the Shostak topology on X if it satisfies the following conditions:

1. $\tau(\tilde{\Phi}) = \tau(\tilde{X}) = 1$,
2. $\tau(F_E \cap G_E) \geq \tau(F_E) \wedge \tau(G_E), \forall F_E, G_E \in SS(X_E)$
3. $\tau\left(\bigcup_i (F_i)_E\right) \geq \bigwedge_i \tau((F_i)_E), \forall (F_i)_E \in SS(X_E)$

The triplet (X, E, τ) is called Shostak topological space of soft sets and denoted by $SHTS$.

Let $\{(F_E)\}$ be a family of soft sets and $\bigcup (F_E) = \tilde{X}$. $\forall r \in [0, 1]$ $\tau(F_E) \geq r$ $\alpha = \{(F_E^i)_{i \in I}\}$ is open covering of (X, E, τ^r) soft topological space. For each α

$$nerv(X, \tau^r) = \left(\{nerv\alpha\}_{\alpha \in (F_E^i)}, \{p_\alpha^\beta : nerv\beta \rightarrow nerv\alpha\}_{\alpha \prec \beta} \right) \quad (1)$$

be the inverse system of simplicial complexes.

Let Z be the group of integers. If we apply the homology functor H_q for the inverse system (1), then we obtain the following inverse system of groups:

$$H_q(nerv(\tilde{X}, \tau^r); Z) = \left(\{H_q(nerv\alpha, Z)\}_{\alpha \in (F_E^i)}, \{H_q(p_\alpha^\beta)\}_{\alpha \prec \beta} \right).$$

Definition 1. The group $H_q\left(\left(\tilde{X}, \tau^r\right); Z\right) = \varprojlim H_q(nerv\alpha; Z)$ is said to be the r -level homology group of Shostak topological space (X, E, τ) . Since $\tau^r \subset \tau^s$ for $r \geq s$, the inverse system (1) is a subsystem of the following system $nerv(\tilde{X}, \tau^s) = \left(\{nerv\alpha\}_{\alpha \in (F_E^i)}, \{p_\alpha^\beta : nerv\beta \rightarrow nerv\alpha\}_{\alpha \prec \beta} \right)$.

Let (X, E, τ) be the Shostak topological space, $F_E \in X_E$ a soft set. If $\tau(F_E) = 1$, we say $(\tilde{X}, F_E)$ as a pair of Shostak topological spaces. The morphism $f : (\tilde{X}, F_E) \rightarrow (\tilde{Y}, G_E)$ is a gp-map if $f : \tilde{X} \rightarrow \tilde{Y}$ such that $f(F_E) \subset G_E$. The pairs of all Shostak topological spaces and their morphisms constitute a category. We denote this category as $SHTS^2$.

Let $(\tilde{X}, F_E)$ be a pair of Shostak topological spaces. For $\forall r \in [0, 1]$, let us take arbitrary as an open covering $\alpha = \{(F_E^i)_{i \in I}\}$ of the space (X, E, τ^r) . Since $\tau(F_E) = 1$, $F_E \in \tau^r$ and the family $\alpha \cap F_E$ is an open covering of soft set F_E over the space (X, E, τ^r) . Then the pair $(\alpha, \alpha \cap F_E)$ is an open covering of pairs of Shostak topological spaces.

Definition 2. The group

$$H_q \left(\left((\tilde{X}, F_E), \tau^r \right); Z \right) = \varprojlim H_q \left(\left(\text{ner}\alpha, \text{ner}(\alpha \cap F_E) \right); Z \right)$$

is said to be the r -level homology group of the pair $(\tilde{X}, F_E)$ and the group $H_q(\tilde{X}, F_E; Z) = \varinjlim H_q \left(\left((\tilde{X}, F_E), \tau^r \right); Z \right)$ is said to be the homology group of the pair $(\tilde{X}, F_E)$.

It is obvious that $H_q : SHTS^2 \rightarrow \text{Group}$ is a functor.

For the pair $(\tilde{X}, F_E)$, let $i : F_E \rightarrow \tilde{X}$, $j : \tilde{X} \rightarrow (\tilde{X}, F_E)$ be the embedding maps. For every $\alpha \in \{(F_E^i)_{i \in I}\}$, the mappings i, j induces the simplicial mappings

$$i_{\alpha, r} : \text{ner}(\alpha \cap F_E) \rightarrow \text{ner}\alpha, j_{\alpha, r} : \text{ner}\alpha \rightarrow (\text{ner}\alpha, \text{ner}(\alpha \cap F_E)).$$

Theorem 1. Hence for every $\alpha \in \{(F_E^i)_{i \in I}\}$, we obtain the following exact sequence of homology groups:

$$\dots \leftarrow H_q(\text{ner}\alpha, \text{ner}(\alpha \cap F_E)) \leftarrow H_q(\text{ner}\alpha) \leftarrow H_q(\text{ner}(\alpha \cap F_E)) \leftarrow \dots \quad (2)$$

We denote this sequence as $H(\text{ner}_r\alpha)$. If we take the limit, then $\varinjlim_r \varprojlim_\alpha H(\text{ner}_r\alpha)$ is

$$\dots \leftarrow H_q(\tilde{X}; Z) \leftarrow H_q(F_E; Z) \leftarrow H_{q+1}(\tilde{X}, F_E; Z) \leftarrow H_{q+1}(\tilde{X}; Z) \leftarrow \dots$$

Consequently, we obtain a sequence of homology groups of Shostak topological spaces $(\tilde{X}, F_E)$, this sequence is generally not exact.

Theorem 2. Let $(\tilde{X}, F_E)$ be a pair of Shostak topological spaces and (U_E) a soft set such that $\tau(U_E) = 1$. The embedding map $f : (U_E^c, F_E \wedge U_E^c) \rightarrow (\tilde{X}, F_E)$ induces isomorphism of homology groups $f_{*q} : H_q(U_E^c, F_E \wedge U_E^c) \rightarrow H_q(\tilde{X}, F_E)$.

For adding homotopy, define unit interval in *SHTS*. Let (I, τ_e) be unit interval in Euclidean topology. We define Shostak topology (I, τ) by the formula $E = \{*\}$, F_*

$$\tau(F_*) = \begin{cases} 1, & \text{if } F \text{ is open} \\ 0, & \text{if other} \end{cases}$$

and we say (I, τ) as Shostak unit interval. Then for $\forall r \in [0, 1]$, $(I, \tau^r) = (I, \tau)$

Theorem 3. If $f, g : (\tilde{X}, \tau) \rightarrow (\tilde{Y}, \tau')$ are homotopic maps, then $f_{*q} = g_{*q} = H_q(\tilde{X}, \tau; Z) \rightarrow H_q(\tilde{Y}, \tau'; Z)$.

Theorem 4. Assume that $(\tilde{X}, \tau)$ topological spaces are a soft topological space then homology groups of topological spaces are same with homology groups of soft topological spaces.

References

- [1] S. Eilenberg, N. Steenrod, Foundations of Algebraic Topology, Princeton University Press, 1952.
- [2] Sadi Bayramov, Cigdem Gunduz, Taha Yasin Ozturk, Homology Theory in the Category of Fuzzy Topological Spaces: Cech Homology Theory, Lambert Academic Publishing, 2012.
- [3] A. Shostak, On a fuzzy topological structure, Rendiconti Ciecolo Matematico Palermo (Supp. Ser. II) 11 (1985) 89-103.

**COMPUTATIONAL MATHEMATICS AND
MATHEMATICAL MODELING**

**A Polynomial Differential Quadrature–Crank–Nicolson
Scheme for a Reaction–Diffusion Dengue Epidemic
Model**

Rahul Agarwal, Sumita Dahiya

Netaji Subhas University of Technology, New Delhi, India
(e-mail: rahul.agarwal.phd25@nsut.ac.in)

We present the PDQ-CN scheme, a numerical method for the reaction–diffusion model of secondary dengue virus (DENV) infection proposed by Elaiw and Alofi, in which polynomial differential quadrature (PDQ) on Chebyshev–Gauss–Lobatto nodes discretizes the spatial derivatives and a Crank–Nicolson (CN) scheme advances the solution in time; the resulting bilinear reaction terms are linearized using the Rubin–Graves technique, so that each time step reduces to a single monolithic linear solve rather than a nonlinear iteration. The PDQ-CN scheme reproduces all four analytical equilibria of the model to within 0.002% relative error, attains a measured temporal order of 1.997–2.032, and shows spatial errors falling more than eight orders of magnitude by $N = 21$ nodes, which already matches an $N = 161$ reference to within floating-point precision at roughly 1/145th the computational cost.

References

- [1] N. Haider, M. N. Hasan, J. Onyango, et al., Global dengue epidemic worsens with record 14 million cases and 9,000 deaths reported in 2024. *International Journal of Infectious Diseases* **158**, 107940 (2025)
- [2] T. P. Gujarati, G. Ambika, Virus antibody dynamics in primary and secondary dengue infections. *Journal of Mathematical Biology* **69**(6–7), 1773–1800 (2014)
- [3] A. M. Elaiw, A. S. Alofi, Global dynamics of secondary DENV infection with diffusion. *Journal of Mathematics* **2021**, 5585175 (2021)

Computational Analysis of Fluid Flow in Complex Pipeline Networks

Kamil Aida-zade^a, Yegana Ashrafova^{a,b}

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Baku State University, Baku, Azerbaijan*

(e-mail: ashrafova.yegana@gmail.com)

This paper proposes a numerical approach for the simulation and analysis of transient fluid flow in interconnected pipeline networks characterized by complex topological configurations. The flow dynamics are described by a system of hyperbolic partial differential equations that model the unsteady motion of a compressible fluid within each pipeline segment. Special consideration is given to the formulation and numerical treatment of coupling conditions at network junctions, where the continuity and conservation laws based on Kirchhoff-type principles must be satisfied.

The governing equations are discretized using an implicit finite-difference scheme, resulting in systems of algebraic equations for each time step. To efficiently solve these systems over networks with arbitrary connectivity, a generalized sweep algorithm is developed for interconnected pipeline segments.

Consider a pipeline network consisting of several linear segments connected at nodes. The unsteady motion of fluid in each segment is described by a system of hyperbolic partial differential equations representing conservation of mass and momentum. For clarity and simplicity in presenting the numerical schemes, let us consider an oil pipeline consisting of five sections, the structure of which is shown in Fig. 1.

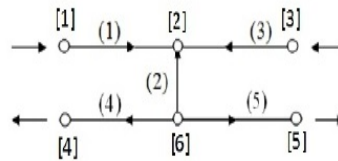


Fig.1. Scheme of the oil pipeline with five sections

The arrows in the figure formally indicate the assumed (not necessarily actual) direction of oil flow.

Let the set of segments be denoted by S : $S = \{s_1, s_2, \dots, s_K\}$ and the set of vertices by V . The numbers in square brackets correspond to the numbers of the vertices (nodes), while the numbers in parentheses denote the segment numbers.

The linearized system of differential equations describing the unsteady isothermal laminar flow of a liquid with constant density ρ in the s -th linear section of an oil pipeline of length l_s and diameter d_s can be written in the following form [1]:

$$\begin{cases} -\frac{\partial P^s(x, t)}{\partial x} = \frac{\rho}{A^s} \frac{\partial Q^s(x, t)}{\partial t} + 2a_s \frac{\rho}{A^s} Q^s(x, t), \\ -\frac{\partial P^s(x, t)}{\partial t} = c^2 \frac{\rho}{A^s} \frac{\partial Q^s(x, t)}{\partial x}, \end{cases} \quad x \in (0, l_s), \quad t \in (0, T], \quad s \in S. \quad (1)$$

where $P^s(x, t)$ and $Q^s(x, t)$ are respectively the pressure and the flow rate of the fluid at time t at point $x \in (0, l_s)$ of the s -th pipeline segment; c is the speed of sound in the medium; and A^s – is the cross-sectional area of the pipeline segment.

Assume that the kinematic viscosity coefficient γ does not depend on pressure; then the corresponding condition $2a = \frac{32\gamma}{d^2} = \text{const}$ is sufficiently accurate for the laminar flow regime.

At internal junctions $v_j \in V$, boundary conditions enforce: mass conservation (sum of inflows = sum of outflows) and pressure continuity across connected segments.

Mathematically, for vertex $v_j \in V$ connected to segments in sets I_j (incoming) and O_j (outgoing):

$$\sum_{i \in I_j} Q^i(v_j, t) - \sum_{o \in O_j} Q^o(v_j, t) = Q_j^{ext}(t), \quad t \in [t_0, T], \quad (2)$$

$$P^i(v_j, t) = P^o(v_j, t), \quad i \in I_j, \quad o \in O_j, \quad t \in [t_0, T], \quad (3)$$

These nonlocal boundary conditions couple the subsystems and make the numerical solution more complex.

At boundary vertices (inlet/outlet points), operational conditions are prescribed either as: Pressure control:

$$P(v_b, t) = P_b(t), \quad t \in [t_0, T]. \quad (4)$$

Flow control:

$$Q(v_b, t) = Q_b(t), t \in [t_0, T]. \quad (5)$$

Due to the extended operation of the pipeline before $[\tau, T]$, i.e. before monitoring begins, the initial state $P^s(x, t_0)$, $Q^s(x, t_0)$ is not exactly known [6]. Instead, it is assumed to belong to a set of admissible initial states defined by parameter $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_r\} \in R$:

$$P^s(x, t_0) = P_s(x; \lambda), Q^s(x, t_0) = Q_s(x; \lambda). \quad (6)$$

The functions $Q_s(x, \lambda)$, $P_s(x, \lambda)$, $\lambda \in R$ are assumed to be given. We will assume that the functions defining the initial state of the process and the boundary conditions satisfy all the conditions required for the existence and uniqueness of the solution of the corresponding boundary value problem.

To numerically solve the formulated boundary value problem (1)-(6) with non-separated boundary conditions, the implicit finite difference method [2] is applied to each pipeline segment. Nonlocal boundary conditions at junctions are handled by a sweeping algorithm [3]. This decouples the global system into smaller local systems that can be solved in parallel, improving computational efficiency for networks with many segments. Numerical experiments are carried out.

The obtained results show that the proposed numerical method provides an effective tool for analyzing transient processes in complex pipeline networks. The use of an implicit grid scheme ensures numerical stability even for relatively large time steps. By appropriately defining the node connectivity and boundary conditions, the same computational framework can be applied to large-scale pipeline systems with arbitrary topology.

References

- [1] I.A. Charny *Unsteady Motion of Real Fluid in Pipes*. Moscow: Nedra, 199 p. (1975).
- [2] A.A. Samarskii, A.V. Gulin. *Numerical Methods*. Moscow: Nauka, 428 p. (1989).
- [3] K.R. Aida-zade, Y.R. Ashrafova *Solving systems of differential equations of block structure with nonseparated boundary conditions* J.of Applied and Industrial Mathem. 9(1), 2015, 1-10.

Analysis of Methods, Models and Algorithms of Mathematical Modeling Used in Logistics Research

Konul Aliyeva^a, Khatira Babayeva^b

^a*Sumgayit State University, Sumgayit, Azerbaijan*

^b*Sumgayit secondary school No. 26 Sumgayit, Azerbaijan*

(e-mail: konul.aliyeva@sdu.edu.az, rkeliyeva78@gmail.com)

The rapid development of the global economy, the intensification of competition, and the limitation of resources make it essential in the modern era to increase the efficiency of logistics systems. In making scientifically justified decisions for optimizing logistics processes, mathematical modeling methods, models, and algorithms play a crucial role.

In particular, the application of mathematical models in areas such as transportation, warehousing, inventory management, and supply chain planning expands the potential to reduce operational costs, minimize time losses, and improve service quality.

The integration of digitalization and big data technologies into logistics introduces new opportunities, as well as new challenges, for data-driven decision-making and process optimization. Therefore, the topic remains highly relevant and carries significant practical importance in modern management systems.

1. The concept of a mathematical model. A mathematical model is a simplified representation of complex reality, in which events and processes are described through mathematical means — such as functions, variables, equations, constraints, and parameters. Mathematical modeling is used to analyze, predict, and optimize the behavior of real logistics systems.

In logistics, mathematical models are generally developed in the following forms:

1. Analytical models – models that represent system behavior through mathematical equations, particularly linear and nonlinear programming models.
2. Stochastic models – models that take into account random variables and probability distributions (for example, in inventory management).
3. Simulation models – models that imitate the operation of a system on a computer, suitable for studying complex and non-deterministic processes.

4. Algorithmic models – models constructed based on specific procedures and computational sequences, such as shortest-path or transportation problems.

The use of such models makes it possible to replace intuition-based decision-making with scientifically grounded approaches, enabling more precise and objective analysis of logistics processes.[1].

2. General linear programming model. The foundation of most static logistics problems lies in *linear programming (LP)*. This model can be expressed in the following general form:

$$\text{Minimize (or Maximize) } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij}$$

subject to:

$$\sum_{j=1}^n a_{ij}x_{ij} \leq b_i, i = 1, 2, \dots, m, x_{ij} \geq 0, \forall i, j$$

where:

1. (x_{ij}) - the quantity of goods transported or allocated from source i to destination j ;
2. (c_{ij}) - the unit cost of that operation (e.g., transportation or production);
3. (b_i) - the available resources or supply capacity of source i ;
4. The objective function (Z) represents the total cost (to be minimized) or total profit (to be maximized).

This formulation provides a mathematical foundation for many logistics optimization problems, especially in areas such as supply allocation, distribution planning, and resource balancing [2].

3.Dynamic models:description and general concept

Dynamic models describe the states of logistics systems that change over time. In these models, the decision variables depend on time (t), and the main objective is to ensure the optimal performance of the system within a specific time interval.

4.Monte Carlo simulation

The Monte Carlo method performs mathematical analysis of stochastic processes through a large number of random samples.

Steps:

1. Identify the probability distributions $P(x_i)$ of the random variables;
2. Generate numerous random realizations $\xi_1, \xi_2, \dots, \xi_n$ from these distributions;
2. Calculate the objective function for each realization $Z_i = f(x, \xi_i)$;
3. Find the average value as the overall result:

$$E[Z] \approx \frac{1}{N} \sum_{i=1}^N f(x, \xi_i)$$

5.Dynamic model for inventory management

In a supply chain, the inventory level management can be represented dynamically as follows:

$$I_{t+1} = I_t + Q_t - D_t$$

Objective function:

$$\text{Minimize } Z = \sum_{t=1}^T (C_h I_t + C_o Q_t)$$

This model is used to determine the optimal ordering times and quantities, so that the total cost of inventory management (including storage and ordering expenses) is minimized.

References

- [1] Daganzo, C. F. *Fundamentals of transportation and traffic operations*. Pergamon. (1997).
- [2] Papageorgiou, M. *Transportation systems engineering: Theory and methods*. Kluwer Academic Publishers. (1991).
- [3] Sheffi, Y. *Urban transportation networks: Equilibrium analysis with mathematical programming methods*. MIT Press.(1985).

Discrete Lyapunov Stability of a 3D Godunov-Type Scheme for Symmetric Quasilinear Hyperbolic Systems with Dissipative Boundary Conditions

Rakhmatillo Alov^a, Dilduza Nematova^b

^a100174, University street 4, Tashkent, Uzbekistan

^b100066, Islam Karimov street 49, Tashkent, Uzbekistan
(e-mail: aloevr@mail.ru)

Abstract. This paper establishes the discrete Lyapunov stability and global exponential decay of a three-dimensional Godunov-type finite-volume difference scheme. For quasilinear symmetric hyperbolic systems with dissipative boundary conditions, we prove energy nonincrease using a fully discrete energy framework.

1. Introduction. Multidimensional quasilinear hyperbolic systems model transport phenomena where discrete stability preservation under nonlinear boundary conditions is a critical challenge. This paper addresses this gap by establishing a direct discrete Lyapunov-based energy mechanism for fully discrete 3D Godunov-type schemes.

2. Continuous and Discrete Problem Setup

2.1. Continuous System

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a piecewise smooth boundary $\partial\Omega \in C^1$, and let $Q_T = \Omega \times (0, T)$. We consider the 3D quasilinear symmetric hyperbolic system:

$$A_0(u) \frac{\partial u}{\partial t} + A_1(u) \frac{\partial u}{\partial x} + A_2(u) \frac{\partial u}{\partial y} + A_3(u) \frac{\partial u}{\partial z} = f(x, y, z, t, u)$$

where $u(x, y, z, t) \in \mathbb{R}^m$ is the unknown vector, $A_k(u) \in \mathbb{R}^{m \times m}$ are symmetric matrices ($A_k = A_k^T$), and $A_0(u)$ is strictly positive definite, satisfying $\langle A_0(u)\xi, \xi \rangle \geq \alpha_0 |\xi|^2$ for $\alpha_0 > 0$.

On the boundary $\partial\Omega$, we impose quasilinear dissipative boundary conditions $B(u)u = 0$, such that the normal flux matrix

$$A_n(u) = \sum_{k=1}^3 n_k A_k(u)$$

(with outward normal n) satisfies the dissipativity inequality:

$$\langle A_n(u)u, u \rangle \leq 0, \quad (x, y, z) \in \partial\Omega$$

For strictly dissipative boundaries, there exists $\delta > 0$ such that $\langle A_n(u)u, u \rangle \leq -\delta|u|^2$, yielding an exponential decay of the continuous energy

$$E(t) = \iiint_{\Omega} \langle A_0(u)u, u \rangle dx dy dz$$

2.2. Fully Discrete 3D Godunov Discretization

We introduce a uniform grid with spatial steps h_x, h_y, h_z and time step τ , defining $u_{ijk}^n \approx u(ih_x, jh_y, kh_z, n\tau)$. The conservative 3D Godunov-type difference scheme is expressed as:

$$\frac{u_{ijk}^{n+1} - u_{ijk}^n}{\tau} + \frac{F_{i+\frac{1}{2},j,k}^n - F_{i-\frac{1}{2},j,k}^n}{h_x} + \frac{G_{i,j+\frac{1}{2},k}^n - G_{i,j-\frac{1}{2},k}^n}{h_y} + \frac{H_{i,j,k+\frac{1}{2}}^n - H_{i,j,k-\frac{1}{2}}^n}{h_z} = f_{ijk}^n$$

where F, G, H represent numerical fluxes computed via localized Riemann solvers. Utilizing the numerical viscosity approach, the interface flux can be decomposed as:

$$F_{i+\frac{1}{2},j,k}^n = \frac{1}{2} (F(u_{ijk}^n) + F(u_{i+1,j,k}^n)) - \frac{1}{2} D_{i+\frac{1}{2},j,k}^x (u_{i+1,j,k}^n - u_{ijk}^n)$$

where $D^x, D^y, D^z \geq 0$

are the numerical diffusion matrices ensuring discrete scheme stability.

3. Discrete Lyapunov Stability Analysis

3.1. Discrete Functional and Main Theorems

We construct the discrete Lyapunov functional (discrete energy) mimicking the continuous framework:

$$E^n = \sum_{i,j,k} \langle A_0(u_{ijk}^n)u_{ijk}^n, u_{ijk}^n \rangle h_x h_y h_z$$

Owing to the positive definiteness of $A_0(u)$, the discrete energy norm satisfies $\alpha_0 \|u^n\|_h^2 \leq E^n \leq \beta_0 \|u^n\|_h^2$.

Theorem 1 (Discrete Lyapunov Stability). *Let the numerical fluxes be consistent, conservative, and monotone, and let the multidimensional Courant-Friedrichs-Lewy (CFL) condition hold:*

$$\tau \left(\frac{\lambda_x^{\max}}{h_x} + \frac{\lambda_y^{\max}}{h_y} + \frac{\lambda_z^{\max}}{h_z} \right) \leq C_{CFL} < 1$$

Assuming sufficiently small initial data $\|u^0\|_h \leq \varepsilon$, the discrete functional satisfies $E^{n+1} \leq E^n$ for all $n \geq 0$, demonstrating that the scheme is nonincreasing and Lyapunov stable.

Theorem 2 (Discrete Exponential Stability). *Under the conditions of Theorem 1, if the discrete boundary flows satisfy a strict dissipativity constraint $\sum_{\partial\Omega_h} \langle \hat{F}^n u_{ijk}^n, u_{ijk}^n \rangle \leq -\delta \sum_{\partial\Omega_h} |u_{ijk}^n|^2$ for $\delta > 0$, then there exist grid-independent constants $\gamma > 0$ and $C > 0$ such that:*

$$E^n \leq CE^0 e^{-\gamma t^n}, \quad t^n = n\tau$$

Conclusion. We established a rigorous fully discrete Lyapunov stability analysis for a 3D Godunov-type scheme under quasilinear dissipative boundary feedback. The validated global energy decay mechanism provides a solid mathematical foundation for extending stable multidimensional solvers to higher-order unstructured meshes.

References

- [1] S.K. Godunov, *A difference method for numerical calculation of discontinuous solutions of equations of hydrodynamics*, Mat. Sbornik, 1959, 47(3), 271-306.
- [2] K.O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math., 1954, 7(2), 345-392.
- [3] P.D. Lax, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math., 1957, 10(4), 537-566.
- [4] H.-O. Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math., 1970, 23(3), 277-298.
- [5] R.J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, Cambridge, 2002.

About the Solution of a Nonlinear Diffusion Equation in Non-Divergence Form With a Source

Mirsaid Aripov, Makhmud Bobokandov

National University of Uzbekistan, Tashkent, 100174, Uzbekistan
(e-mail: mirsaidaripov@mail.ru, m.boboqandov@nuu.uz)

In this paper we consider the Cauchy problem

$$\begin{cases} u_t = u^q \operatorname{div}(|x|^n |\nabla u|^{p-2} \nabla u) + C(T_0 + t)^l u^\beta, & (x, t) \in \mathbb{R}^N \times (0, t_f], \\ u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \quad (1)$$

Where $q \in \mathbb{R} \setminus \{1\}, p \geq 2, n \in (-N, \infty), \beta > 1, l > -1, C, T_0, t_f > 0, N \geq 1,$

$$0 \leq u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

The equation (1) included many known equations, such as the porous medium equation, p-Laplacian equation, heat equation [2], [3], Leibenson equation, Boussinesq equation in filtration of liquid and gas, and so on [1], [5].

We denote $B_R = \{x \in \mathbb{R}^N : |x| < R\}$, $Q_R = B_R \times (0, t_f)$, and the weighted Sobolev space relevant to the diffusion operator is

$$W^{1,p}(B_R; |x|^n) = \{u \in L^p_{\text{loc}}(B_R) : \|\nabla u\|_{L^p(B_R; |x|^n)} < \infty\}.$$

Definition 1. A measurable function $u \geq 0$ is an energy solution of (1) on Q_R if:

1. $u \in L^\infty(0, t_f; L^r(B_R)) \cap L^p(0, t_f; W^{1,p}(B_R; |x|^n))$ for some $r \geq \max\{1, 1-q\}$;
2. $u^{1-q} \in L^1(Q_R)$ and $|x|^n |\nabla u|^{p-1} \in L^1(Q_R)$;
3. for every $\varphi \in C_c^\infty(B_R \times [0, t_f])$:

$$\begin{aligned} - \int_{Q_R} \frac{u^{1-q}}{1-q} \varphi_t dx dt &= \int_{B_R} \frac{u_0^{1-q}}{1-q} \varphi(\cdot, 0) dx - \int_{Q_R} |x|^n |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx dt \\ &\quad + C \int_{Q_R} (T_0 + t)^l u^{\beta-q} \varphi dx dt. \end{aligned}$$

Theorem 1. *Let $r \geq \max\{1, 1 - q\}$.*

1. Subcritical ($\beta < \beta_c$): *If $\|u_0\|_{L^{s_c}(\mathbb{R}^N)}$ is sufficiently small, then the energy solution of (1) satisfies $\sup_{t>0} \|u(\cdot, t)\|_{L^r(\mathbb{R}^N)} < \infty$ for every r .*
2. Supercritical ($\beta > \beta_c$): *For any nonnegative nontrivial $u_0 \in L^1 \cap L^\infty$, there exist initial amplitudes $A_* > 0$ such that for $u_0 = A\phi$ with $A > A_*$ the regularized solution satisfies $\|u^\varepsilon(\cdot, t)\|_{L^\infty(B_R)} \rightarrow \infty$ in finite time (blow-up within $[0, t_f]$).*
3. Critical ($\beta = \beta_c$): *Every nonnegative nontrivial solution of the regularized problem blows up; no global bound of the form 1 holds for any nontrivial u_0 .*

Where $s_c = 1 + \frac{N(\beta - \beta_c)}{(1+l)(p-n)}$ and $\beta_c = 1 + (1+l)(q + p - 2 + \frac{p-n}{N})$ - the critical Fujita exponent [4].

Acknowledgment. The present work partially supported by grant “Mathematical modelling of processes described by nonlinear, divergent and non-divergent parabolic equations and systems” AL-9224104601 from the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan.

References

- [1] A. Samarskii, V. Galaktionov, S. Kurdyumov, and A. Mikhailov, *Blow-Up in Quasilinear Parabolic Equations*, Walter de Grueter, Berlin, 1995.
- [2] M. Aripov, M. Bobokandov, and M. Mamatkulova, *Analysis of a double nonlinear diffusion equation in inhomogeneous medium*, Journal of Mathematical Sciences, 2025, 289(5), 657-669.
- [3] M. Aripov, M. Bobokandov, *Analysis of a double nonlinear parabolic equation with a source in an inhomogeneous medium*, Proceedings of Institute of Mathematics & Mechanics National Academy of Sciences of Azerbaijan, 2024, 50(2), 285-306.
- [4] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$* , Journal of the Faculty of Science, the University of Tokyo, Section 1 A, Mathematics, 1966, 13, 109-124.

- [5] M. Aripov, M. Bobokandov, and M. Mamatkulova, *To numerical solution of the non-divergent diffusion equation in non-homogeneous medium with source or absorption*, AIP Conference Proceedings, 2024, 3085(1), 020024.

Mathematical Modeling of Nonlinear Biological Populations

Mirsaid Aripov^a, Firuza Kabiljanova^a, Maftuha Sayfullayeva^b

^a*National University of Uzbekistan, Tashkent, Uzbekistan*

^b*Tashkent University of Information Technologies, Tashkent, Uzbekistan*

(e-mail: mirsaidaripov@mail.ru, kabiljanovafiruza@mail.ru,
maftuha87@mail.ru)

In recent years, various aspects of biological populations have been intensively studied by many authors [3-9]. The first mathematical models of biological populations were studied in [1-2], where the following estimate was established $c \geq c_{min} = 2\sqrt{kD}$ for a speed c of wave type solution for the following semi linear parabolic equation.

$$u_t = Du_{xx} + ku(1 - u)$$

where scalar function $u(x, t)$ is subject to specified initial and boundary conditions, and k, D are positive constants. This equation historically has been studied as a model for the spread of a favorable gene in a diploid population, originally introduced by R.A. Fisher (see [1], [2]). A similar heuristic conclusion, based on genetics, was also proposed by A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov in [2], which provided the foundation for a more rigorous analytical approach to Fisher's equation. In this work, we consider the following Cauchy problem for a doubly nonlinear parabolic equation with variable density in nondivergent form in the domain $Q = (t > 0, \mathbb{R}^N)$

$$|x|^{-l} \frac{\partial u}{\partial t} = u^n \nabla (D|x|^q u^{m-1} |\nabla u^k|^{p-2} \nabla u) + k_1(t) |x|^{-l} u(1 - u^\beta), \quad \beta \geq 1, \quad (1)$$

$$u(0, x) = u_0(0, x), \quad x \in \mathbb{R}^N. \quad (2)$$

is considered. This mathematical model, particularly for specific values of the numerical parameters characterizing the nonlinear medium, can be used to model biological populations, processes such as viral spread, nonlinear diffusion, heat conduction, and the filtration of liquids and gases (see [5, 6] and references therein). Based on an approximate self-similar approach and the comparison principle, we prove global solvability, obtain Fujita-type estimates for both the solutions and the free boundary (front), and establish properties such as the finite speed of perturbation and spatial localization. Additionally, the problem of finding an initial approximation is solved for the slow diffusion, fast diffusion, and critical cases. Finally, estimates for the solution for the slow diffusion case are established. This mathematical model, particularly for specific values of the numerical parameters characterizing the nonlinear medium, can be used to model biological populations, processes such as viral spread, nonlinear diffusion, heat conduction, and the filtration of liquids and gases (see [5, 6] and references therein).

Based on an approximate self-similar approach and the comparison principle, we prove global solvability, obtain Fujita-type estimates for both the solutions and the free boundary (front), and establish properties such as the finite speed of perturbation and spatial localization. Additionally, the problem of finding an initial approximation is solved for the slow diffusion, fast diffusion, and critical cases. Finally, estimates of the solution for the slow diffusion case are established. Using the replacements

$$u(t, x) = \bar{u}(t) w(\tau(t), \varphi(x)),$$

where

$$\tau(t) = \int [\bar{u}(t)]^{k(p-2)+m+n-1} dt,$$

and

$$\varphi(x) = \frac{p-q-l}{p} |x|^{\frac{p}{p-q-l}}, \quad \text{if } p > q+l,$$

For the slow diffusion case $k(p-2) + m + n - 1 > 0$, the fast diffusion case $k(p-2) + m + n - 1 < 0$, and the critical case $k(p-2) + m + n - 1 = 0$, it is easy to verify that equation (1) is transformed into the following radially

symmetric form:

$$\frac{\partial w}{\partial \tau} = w^n \varphi^{1-s} \frac{\partial}{\partial \varphi} \left(D \varphi^{s-1} w^{m-1} \left| \frac{\partial w^k}{\partial \varphi} \right|^{p-2} \frac{\partial w}{\partial \varphi} \right) - b(\tau) w^\beta, \quad (3)$$

$$w(0, x) = w_0(0, x), \quad x \in \mathbb{R}^N.$$

where

$$s = \frac{p(N-l)}{p-q-l}, \quad N > l, \quad p > q+l,$$

and the function $b(\tau)$ is defined by

$$b(\tau) = k_1(t) [\bar{u}(t)]^{\beta-k(p-2)-m-n+1}.$$

The reduction of the initial problem (1) to (3) makes it possible to study the qualitative properties of problem (1) more easily. In the case $k(p-2)+m+n-1 > 0$, using the self-similar approach [6] for the principal part of equation (3), we construct the following Zeldovich–Barenblatt type solution

$$u(t, x) = (T+t)^{-\frac{s}{p+(k(p-2)+m+n)s}} (a - \xi^{p/(p-1)})_+, \quad \xi = \varphi(|x|) [\tau(t)]^{-1/p}, \quad a > 0$$

where

$$\varphi(|x|) = \frac{p-q-l}{p} |x|^{\frac{p}{p-q-l}}, \quad p > q+l,$$

with the property of finite-speed propagation of the population.

Based on the constructed Zeldovich–Barenblatt type solution, the finite-speed propagation property and spatial localization of the solution are established.

$$u(t, x) = (T+t)^{-s/p} \exp \left[-k^{p-2} \left(\frac{\xi}{p} \right)^{\frac{p}{p-1}} \right], \quad s = \frac{p(N-l)}{p-q-l}, \quad N > l, \quad p > q+l,$$

$$u(t, x) = (T+t)^{-1/p} \exp \left[-k^{p-2} \left(\frac{\xi}{p} \right)^{\frac{p}{p-1}} \right], \quad \text{if } p = q+l.$$

$$\xi = \frac{\varphi(x)}{(T+t)^{1/p}}, \quad \varphi(|x|) = \frac{p-q-l}{p} |x|^{\frac{p}{p-q-l}}, \quad \text{if } p > q+l,$$

$$\varphi(|x|) = \ln(|x|), \quad \text{if } p = q + l.$$

The following estimate for the wave speed $c(t)$ is established

$$c(t) = \frac{d|x(t)|}{dt} \cong 2 [k_1 D k^{p-2}]^{1/p} t^{\left(\frac{2}{p}-1\right)},$$

It is shown that the Malthus population model described by equation (1), by constructing the Zeldovich–Barenblatt type exact solution, possesses the finite-speed propagation property and spatial localization. The numerical analysis of the considered problem is also discussed.

References

- [1] R.A. Fisher, *The Wave of Advance of Advantageous Genes*, Annals of Eugenics, 1937, 7, 355–369.
- [2] A.N. Kolmogorov, I.G. Petrovskii, N.S. Piskunov, *A Study of the Diffusion Equation with Increase in the Amount of Substance, and its Application to a Biological Problem*, Kluwer Academic Publishers, 1991, pp. 248–270.
- [3] M. Ughi, *A Degenerate Parabolic Equation Modeling Spread of an Epidemic*, Annali di Matematica Pura ed Applicata, 1986, 143, 385–400.
- [4] J.D. Murray, *Mathematical Biology*, 2nd ed., Springer, New York, 1993.
- [5] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, A.P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin, 1995.
- [6] M. Aripov, Sh. Sadullaeva, *Computer Modeling of Nonlinear Diffusion Processes*, National University of Uzbekistan, Tashkent, 687 pp.
- [7] M.M. Aripov, S.A. Sadullayeva, M.Z. Sayfullayeva, *To Mathematical Modeling of Nonlinear Problem Biological Population in Nondivergent Form with Variable Density*, AIP Conference Proceedings, 2021, 2325(1), 020064.
- [8] N.V. Belotelov, A.I. Lobanov, *Population Models with Nonlinear Diffusion*, Mathematical Modeling, 1997, 9(12), 43–55.

- [9] M.M. Aripov, M.Z. Sayfullayeva, *About New Nonlinear Properties of the Problem of Nonlinear Thermal Conductivity*, Journal of Mathematical Modeling, December 2019, 19-26.

Mathematical Modeling of Coupled Diffusion Systems With Convective Transfer

Mirsaid Aripov, Dilobar Nigmanova, Mohidil Rahmonova

National university of Uzbekistan, Tashkent, Uzbekistan
(e-mail: dill198912@gmail.com)

Consider the following Cauchy problem in the domain $Q = \{(t, \mathbf{x}) : t > 0, \mathbf{x} \in \mathbb{R}^N\}$ for the double nonlinear degenerate parabolic systems.

$$\frac{\partial u}{\partial t} = u^n \operatorname{div} (v^{m_1-1} |\nabla u^k|^{p-2} \nabla u) + \operatorname{div}(c(t)u) + \gamma_1(t)u^{p_1}v^{q_1}, \quad (1)$$

$$\frac{\partial v}{\partial t} = v^n \operatorname{div} (u^{m_2-1} |\nabla v^k|^{p-2} \nabla v) + \operatorname{div}(c(t)v) + \gamma_2(t)u^{p_2}v^{q_2}.$$

$$u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N. \quad (2)$$

where numbers $n > 0$, p_i , q_i , p , m_i , $i = 1, 2$ are the given numerical parameters, $0 \leq u_0(x)$, $v_0(x) \in C(\mathbb{R}^N)$, the function $0 < c(t)$ is the speed of convective transfer, $0 < \gamma_i(t) \in C(0, \infty)$. The system of equations of non-divergent form is often used to describe various physical phenomena, such as the diffusion process for biological species, resistive diffusion phenomena in magnetic fields without force, flux shortening, the spread of infectious diseases, etc. (see [1–9] and citations therein).

The system (1) in the domain where $u(t, x) \equiv 0$ and $v(t, x) \equiv 0$ are degenerate type parabolic systems. Therefore, for the solution of the system (1) may have place phenomena of the *finite speed of a propagation* (FSP), i.e., there are the functions $l_1(t)$, $l_2(t)$, that $u(t, x) \equiv 0$ and $v(t, x) \equiv 0$ when $|x| \geq l_1(t)$ and $|x| \geq l_2(t)$. In the case $l_1(t)$, $l_2(t) < \infty$ for $\forall t > 0$ solution of the problem (1) is called a *space localization of a perturbation* (SLP). The surfaces $|x| = l_1(t)$ and $|x| = l_2(t)$ are called a free boundary or a front.

A nonlinear splitting algorithm [1] is used to reduce system (1) to a self-similar form.

First, we solve the auxiliary system

$$\frac{d\bar{u}}{dt} = \bar{v}^{q_1}, \quad \frac{d\bar{v}}{dt} = \bar{u}^{q_2}. \quad (3)$$

which admits solutions of the form

$$\bar{u}(t) = A_1(T + t)^{n_1}, \quad \bar{v}(t) = A_2(T + t)^{n_2}, \quad T > 0. \quad (4)$$

Here, the functions $\bar{u}(t)$ and $\bar{v}(t)$ represent the contributions of the source or absorption terms to the solution of system (1).

Using the solutions (4), we introduce the transformation

$$u(t, x) = \bar{u}(t) w(\tau(t), x), \quad v(t, x) = \bar{v}(t) z(\tau(t), x), \quad (5)$$

where $\tau(t)$ is a new time variable determined by the nonlinear splitting algorithm.

Furthermore, we seek a self-similar solution in the form

$$w(\tau(t), x) = f_1(\xi), \quad z(\tau(t), x) = f_2(\xi), \quad (6)$$

where $\xi = |x| \tau^{-1/p}$.

Substituting (5) and (6) into system (1), we obtain the following self-similar system:

$$\begin{aligned} f_1^n \xi^{1-N} \frac{d}{d\xi} \left(\xi^{N-1} f_1^{m_1-1} \left| \frac{df_1^k}{d\xi} \right|^{p-2} \frac{df_2}{d\xi} \right) + \frac{\xi}{p} \frac{df_1}{d\xi} \\ + \frac{\alpha_1}{1 - (k(p-2) + m_1 - 1) \alpha_1} (f_1 + \varepsilon f_1^{p_1} f_2^{q_1}) = 0, \\ f_2^n \xi^{1-N} \frac{d}{d\xi} \left(\xi^{N-1} f_2^{m_2-1} \left| \frac{df_2^k}{d\xi} \right|^{p-2} \frac{df_1}{d\xi} \right) + \frac{\xi}{p} \frac{df_2}{d\xi} \\ + \frac{\alpha_2}{1 - (k(p-2) + m_2 - 1) \alpha_2} (f_2 + \varepsilon f_1^{p_2} f_2^{q_2}) = 0. \end{aligned} \quad (7)$$

For computational goal it is important to investigate the qualitative properties of solutions such as global solvability, the condition of arising blow up

solution, properties of finite speed of perturbation, space localization, estimate, asymptotic of solution, to find an exact solution (if it possible) depending on the value of numerical parameters of the considered problem.

In this talk, we demonstrate the capabilities of self-similar and approximately self-similar approaches for studying nonlinear reaction–diffusion systems with time-dependent convective transfer and a source. The influence of the system parameters on the evolution of the process is investigated. Estimates of weak solutions are established for the cases of slow diffusion, fast diffusion, and the critical case, corresponding to the conditions $k(p-2)+m_i+n-1 > 0$, $k(p-2)+m_i+n-1 < 0$, $k(p-2)+m_i+n-1 = 0$, respectively. The asymptotic behavior of the self-similar solution is analyzed. It is shown that finite speed of propagation, localization of solutions, and finite-time blow-up occur for certain parameter values. Numerical results illustrating the theoretical analysis are also presented.

References

- [1] A.A. Samarskii, S. P. Kurdyumov., V. A. Galaktionov, A. P. Mikhailov *Blow up in quasilinear parabolic equations* M.,1986 437 p.
- [2] Wenshu Zhou, Zhouquo Wu, *Some results on a class of degenerate parabolic equations not in divergence form*, Nonlinear Analysis: Theory, Methods Appl. Vol. 60(5), 2005, pp. 863-886
- [3] Zhou Wenshu, Yao Zhenggen, *Cauchy problem for a degenerate parabolic equation with non-divergence form*, Acta Mathematica Scientia 2010, 30B(5): 1679-1686.
- [4] H. Fujita, *On the blowing up of solutions of the Cauchy problem* Journal of the Faculty of Science University of Tokyo A, vol. 16, pp. 105–113, 1966.
- [5] Maura Ughi, *A degenerate parabolic equation modeling spread of an epidemic*, Ann. Mat. Pura Appl. 143 (1986) pp. 385-400.
- [6] Aripov M. M., Sadullaeva Sh. A. *An asymptotic analysis of a self-similar solution for the double nonlinear reaction-diffusion system*, Nanosystems-Physics Chemistry Mathematics, vol. 6, 6, 2015, pp. 793-802

- [7] Aripov M. Sadullaeva Sh. A. *Computer modeling of nonlinear diffusion processes*, Tashkent, University, 2020, 670 p.
- [8] M. Aripov, M., Bobokandov, M. M. *Cauchy problem for the double non-linear parabolic equation not in divergent form with a time-dependent source or absorption*, Carpathian Mathematical Publications., 2025, 2

High-Accuracy Difference Schemes For A Multidimensional Parabolic Equation

Mirsaid Aripov^a, Dauletbay Utebaev^b, Sapura Kalmuratova^b

^aNational University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan

^bKarakalpak State University named after Berdakh, Nukus, Uzbekistan
(e-mail: mirsaidaripov@mail.ru, dutebaev_56@mail.ru, kalmuratova.sapura@mail.ru)

The construction of high-accuracy difference schemes with minimal stencils is a priority area in modern numerical methods theory. The numerical study of mathematical models of complex nonstationary processes, such as nonstationary parabolic equations, Sobolev-type pseudoparabolic equations, convection-diffusion equations, etc., occupies a special place. Initial-boundary value problems for these equations arise in the study of environmental problems related to the description of pollutant spread in the atmosphere and water bodies, and the modeling of groundwater contamination. This paper considers a multidimensional parabolic equation with constant coefficients:

$$\frac{\partial u}{\partial t} = Lu + f(x, t), \quad u(x, 0) = u_0(x), \quad u|_{\Gamma} = \mu(x, t), \quad (1)$$

$Lu = \sum_{k=1}^p \frac{\partial^2 u}{\partial x^k} —$ is the p dimensional Laplace operator.

To approximate the spatial variables of problem (1), a high-accuracy finite difference method is used [1]. As a result, we obtain the Cauchy problem for a system of large-dimensional ordinary differential equations:

$$\bar{D} \frac{du_h(t)}{dt} + \bar{A}u_h(t) = \bar{f}_h(t), \quad u_h(0) = u_{h,0}, \quad (2)$$

where $\bar{f} = \left(f + \sum_{\alpha=1}^p \frac{h_\alpha^2}{12} \Lambda_\alpha f \right)^{j+1/2}$, $u_h(t) \in H_h$ approximates $u(x, t)$ at fixed nodes $x = (i_1 h_1, i_2 h_2, \dots, i_p h_p)$, $\bar{D} \equiv D + \sum_{\alpha=1}^p \frac{h_\alpha^2}{12} \Lambda_\alpha$, $\bar{A} = A - \sum_{\alpha=1}^p \frac{h_\alpha^2}{12} \sum_{\beta \neq \alpha} \Lambda_\alpha \Lambda_\beta$, $D \equiv E$, $A = -\Lambda$, $\Lambda = \sum_{\alpha=1}^p \Lambda_\alpha$, $\Lambda_\alpha u_{h_\alpha} = u_{h_\alpha, \bar{x}_\alpha x_\alpha}$, $A \in H_h$. Here, $H_h = \overset{\circ}{W}_2^1(\bar{\omega}_h)$ - is a subspace of discrete functions u_{h_α} , approximating H Hilbert space.

The initial and boundary conditions are approximated exactly.

To approximate problem (2), a piecewise polynomial finite element method of fourth-order accuracy is used [2]:

$$\bar{D}y_t - (\tau^2/12)\bar{A}\dot{y}_t + \bar{A}y^{(0.5)} = \varphi_1, \quad \gamma\bar{D}\dot{y}_t + \alpha\bar{A}y_t + \beta\bar{A}\dot{y}^{(0.5)} = \varphi_2, \quad (3)$$

$$y^0 = u_{h,0}, \quad \dot{y}^0 = \dot{u}_{h,0},$$

where $y \in H_h$, $\varphi_1 = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \bar{f}(t) dt$, $\varphi_2 = \frac{12}{\tau^3} \int_{t_n}^{t_{n+1}} \bar{f}(t) \left(s_1 \vartheta_2^{(1)} + s_2 \vartheta_2^{(3)} \right) dt$, $s_1 = 15\gamma - 35\alpha/3$, $s_2 = 140\gamma - 350\alpha/3$, $\vartheta_2^{(1)} = t - (t_n + \tau/2)$, $\vartheta_2^{(3)} = \tau(\xi^3 - 3\xi^2/2 + \xi/2)$, $\xi = (t - t_n)/\tau$, $\dot{u}_{h,0} = \bar{D}^{-1}(f_h^0 - Au_{h,0})$.

The parameters of scheme (3) obey the fourth-order approximation condition $\alpha + \beta = \gamma$, $\alpha, \beta, \gamma = O(\tau^2)$.

As a result of the study, schemes of fourth-order accuracy in all variables, i.e., $\|y - u\| = O(|h|^4 + \tau^4)$, were constructed. Stability conditions and a priori estimates for the solution in the class of smooth solutions were obtained. Convergence and accuracy theorems were proven. Based on the proposed algorithm, numerical results were obtained on test problems, confirming theoretical conclusions. The numerical results are compared with those of known methods, and the advantages of the constructed numerical algorithms are indicated.

References

- [1] A.A. Samarskii, *High-Order Accuracy Schemes for the Parabolic Heat Conduction Equation*, Journal of Computational Mathematics and Mathematical Physics, 1963, 3(5), 812-840.
- [2] D. Utebaev, M.N. Moskalkov, *Numerical Modeling of Nonstationary Processes in Continuum Mechanics*, Tashkent: Fan va Technologiya, 2012, 160p.

Numerical Study of the Gilson–Pickering Equation for Nonlinear Plasma Wave Dynamics

Sumita Dahiya, Suhana Chaudhary, Priyanka Yadav

*Department of Mathematics, Netaji Subhas University of Technology,
Delhi-110078, India*

(e-mail: sumita.dahiya@nsut.ac.in)

This study investigates the Gilson–Pickering equation governing the nonlinear propagation of waves in ionized gas media and presents a robust mathematical framework capable of describing different nonlinear regimes. The wave propagation process is influenced by the combined effects of nonlinear convection, dispersion, and dissipation arising from charged particle interactions, collisional, and inertial effects. The interaction of charged particles in such media governs the formation and propagation of localized and oscillatory wave patterns. To better understand the associated nonlinear wave dynamics, a phase plane analysis is presented to investigate the equilibrium structure and qualitative behavior of the resulting dynamical system. The accurate simulation of such nonlinear wave phenomena demands reliable and efficient numerical schemes. The proposed approach obtains the numerical solution of the governing equation by combining a Crank–Nicolson time semi-discretization with an improvised cubic B-spline spatial collocation method. The conserved quantities are further examined to evaluate the stability and physical consistency of the proposed numerical approach. The numerical results confirm the accuracy and stability of the approach and demonstrate its effectiveness in capturing nonlinear plasma wave propagation dynamics.

Keywords: Nonlinear dispersive waves, ionized gas media (plasma), Gilson–Pickering equation, wave dynamics, Fourier spectral analysis

References

- [1] Dahiya, S., Yadav, P., Malik, N. (2026). Integrating Mathematical Modeling and Oncology: An Improvised Collocation Approach for Prediction of Chemotherapy Response. *Bulletin of Mathematical Biology*, 88(6), 103.

- [2] Dahiya, S., Malik, N., Yadav, P. (2026). Numerical simulation of solitons and peakons in shallow water wave models using improvised collocation method. *Physica Scripta*, 101(12), 125204.
- [3] Malik, N., Singh, A., Dahiya, S. (2025). An improvised cubic B-spline-based collocation approach for solving nonlinear KG equations. *Numerical Algorithms*, 1-26.
- [4] Yadav, P., Dahiya, S. (2026). Fourth-Order Convergent Scheme for Kudryashov–Sinelnikov Equation in Bubbly Liquids. *Physics Letters A*, 131585.

Local Stability and the Emergence of Stable Two-Cycles in a Nonlinear Difference System

Serbun Ufuk Değer

Kastamonu University, Kastamonu Vocational School, Kastamonu, Türkiye
(e-mail: sudeger@kastamonu.edu.tr)

We investigate the discrete-time tumor-immune interaction model

$$\begin{cases} X_{n+1} = X_n + rX_n \left(1 - \frac{X_n}{K}\right) - \frac{\alpha X_n Y_n}{1 + \beta X_n}, \\ Y_{n+1} = Y_n + \frac{sX_n Y_n}{1 + \beta X_n} - \delta Y_n, \end{cases} \quad (1)$$

where X_n and Y_n denote tumor burden and effector immune-cell level. We assume $r, K, \alpha, s > 0$, $\beta \geq 0$, and $0 < \delta < 1$. A Lyapunov-type combination, $V_n = X_n + (\alpha/s)Y_n$, gives boundedness for every trajectory that remains nonnegative.

The equilibria are $E_0 = (0, 0)$, $E_1 = (K, 0)$, and

$$E^* = (X^*, Y^*), \quad X^* = \frac{\delta}{s - \delta\beta}, \quad Y^* = \frac{rs(K - X^*)}{\alpha K(s - \delta\beta)}.$$

The coexistence equilibrium is biologically admissible exactly when

$$E^* > 0 \iff \frac{sK}{1 + \beta K} > \delta. \quad (2)$$

Moreover, E_0 is always unstable. At E_1 , the eigenvalues are $1 - r$ and $1 + sK/(1 + \beta K) - \delta$. Resolving the boundary $r = 2$ by the second iterate on the invariant line $Y = 0$ gives

$$E_1 \text{ is LAS} \iff 0 < r \leq 2, \quad \frac{sK}{1 + \beta K} < \delta. \quad (3)$$

Thus, existence of E^* and immune-direction stability of E_1 are separated by the same threshold with opposite strict inequalities.

For E^* , let $D = s - \delta\beta$, $A = K - 2X^*$, and $B = D(K - X^*)/s$. The Jury–Schur criterion yields

$$A < (1 - \delta)B, \quad 0 < r < r^*, \quad r^* = \frac{4K}{(2 - \delta)B - 2A},$$

with positive denominator under the first inequality. For

$$K = 10, \quad \alpha = 0.1, \quad \beta = 0.02, \quad s = 0.087, \quad \delta = 0.6,$$

we obtain $X^* = 8$ and $r^* = 580/209 \approx 2.7751$. At $r = r^*$, the multipliers are -1 and $179/209$, the transversality derivative satisfies $b_1 \approx -0.7766 < 0$, and the cubic normal-form coefficient is $l_1 \approx 0.01168 > 0$. Hence a locally stable small-amplitude period-two orbit emerges for $r > r^*$, so the loss of stability is a non-degenerate supercritical flip bifurcation. The critical equilibrium itself is nonhyperbolically locally asymptotically stable, and therefore

$$E^*(r) \text{ is LAS} \iff 0 < r \leq \frac{580}{209}. \quad (4)$$

Direct numerical continuation and exact Jacobian evaluation of the period-two orbit confirm the analytical classification.

References

- [1] V.A. Kuznetsov, I.A. Makalkin, M.A. Taylor, A.S. Perelson, *Nonlinear dynamics of immunogenic tumors: parameter estimation and global bifurcation analysis*, Bulletin of Mathematical Biology, 56(2) (1994), 295–321.
- [2] M.S. Khan, M. Samreen, M.A. Khan, M. De la Sen, *A dynamically consistent nonstandard difference scheme for a discrete-time immunogenic tumors model*, Entropy, 24(7) (2022), 949.

- [3] I. Jena, P. Gupta, A. Priyadarshi, *A discrete-time tumour-immune interaction model: bifurcation and optimal control analysis*, Journal of Difference Equations and Applications, 31(12) (2025), 1514–1545.
- [4] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 3rd ed., Springer, New York, 2004.

Qualitative Properties of Solutions to a Parabolic Heat Equation in Nondivergence Form with Two Nonlinear Source Terms

Oybek Djabbarov, Mahbuba Samadova, Muzaffar Karimov

Karshi State University, Karshi, Uzbekistan
(e-mail: zhabbarov.or@qarshidu.uz)

Consider in $Q = \{(x, t) : x \in R, t > 0\}$ the following Cauchy problem:

$$|x|^{\alpha_1} \frac{\partial u}{\partial t} = u^\beta \frac{\partial}{\partial x} \left(|x|^{\alpha_2} u^{m-1} \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right) + |x|^{\alpha_1} u^q \quad (1)$$

$$u(x, t)|_{t=0} = u_0(x) \geq 0, \quad x \in R. \quad (2)$$

Here the parameters satisfy $\beta < 1$, $m \geq 1$, $p \geq \max\{2, \alpha_2 - \alpha_1\}$, $q > 1$, $\alpha_1 < 0$, $\alpha_2 < 0$ and the initial function $u_0(x)$ is assumed to be continuous, nonnegative, and bounded. It is well known that equation (1) is degenerate, and classical solutions may fail to exist in the cases $u(x, t) = 0$ or $|u_x| = 0$. Therefore, solutions of the Cauchy problem (1)-(2) are understood in the sense of generalized solutions defined via integral identities. In particular, we require

$$0 \leq u, \quad |x|^{\alpha_2} u^{m-1} \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \in C(R \times (t, +\infty))$$

and equation (1) is satisfied in the integral sense. Depending on the values of the parameters, problem (1)-(2) arises in the mathematical modeling of heat transfer, nonlinear diffusion, chemical reactions, and biological processes. [1]-[4].

First, we introduce the following transformation in equation (1):

$$y = u^{1-\beta}. \quad (3)$$

Under this, the Cauchy problem (1)-(2) can be rewritten in the form

$$(1-\beta) L(y) = y^{\frac{\beta}{1-\beta}} \times \\ \times \left[|x|^{\alpha_1} y_t - \frac{\partial}{\partial x} \left(|x|^{\alpha_2} y^{\frac{m}{1-\beta}-1} \left| \frac{\partial y^{\frac{1}{1-\beta}}}{\partial x} \right|^{p-2} \frac{\partial y}{\partial x} \right) + (1-\beta) |x|^{\alpha_1} y^{\frac{q-\beta}{1-\beta}} \right] = 0 \quad (4)$$

with the initial condition

$$y|_{t=0} = y_0(x) = [u_0(x)]^{1-\beta}, \quad x \in R. \quad (5)$$

Theorem 1. Suppose that at least one of the following conditions holds:

$$\alpha_1 < -1, \mu_1 > \mu_2, \text{ or } \alpha_1 = -1, \mu_1 > 1,$$

and assume that the inequality

$$y(x, t) \leq y_+(x, t), \quad x \in R,$$

is satisfied. Then the Cauchy problem (4)–(5) admits a global generalized solution in the domain Q and this solution satisfies the estimate

$$y(x, t) \leq y_+(x, t) = t^{-n_1} \bar{g}(\eta). \quad (6)$$

Here, $\mu_1 = \frac{q-\beta}{1-\beta}$, $\mu_2 = \frac{m+p-2}{1-\beta} + \frac{p-\alpha_2+\alpha_1}{\alpha_1+1}$, $\bar{g}(\eta) = A(a - \eta^{\gamma_1})_+^{\gamma_2}$, $\gamma_1 = \frac{p-\alpha_2+\alpha_1}{p-1}$, $\gamma_2 = \frac{(p-1)(1-\beta)}{m-p+\beta+1}$, $a \geq 0$, $\eta = \frac{|x|}{t^{n_2}}$, $n_1 = \frac{1-\beta}{q-1}$, $n_2 = \frac{p+q-(\beta+m+2)}{(q-1)(p-\alpha_2+\alpha_1)}$ and $\mu_2 -$ is referred to as the Fujita-type critical exponent.

This paper investigates the properties of generalized solutions to a parabolic heat conduction equation in no divergent form with double nonlinear source terms. The model incorporates nonlinear diffusion with variable density and nonlinear source functions, and for cases where classical solutions do not exist, the concept of generalized solutions in the sense of integral identities is employed. By applying an appropriate change of variables and the method of

self-similar solutions, the original nonlinear parabolic equation is reduced to an ordinary differential equation. In the slow diffusion regime, the global existence of solutions is established using the comparison principle, and a Fujita-type critical exponent is determined. The obtained results have significant theoretical importance for the numerical analysis of nonlinear models arising in heat transfer, chemical processes, and biological systems.

References

- [1] Samarskii A. A., Galaktionov V. A., Kurdyumov S. P., Mikhailov A. P. *Blow-Up in Quasilinear Parabolic Equations*. De Gruyter, 1995.
- [2] Aripov M., Sadullaeva S. *Computer simulation of nonlinear diffusion processes*. Tashkent: National University Press, 2020.
- [3] Aripov M., Bobokandov M. The Cauchy problem for a doubly nonlinear parabolic equation with variable density and nonlinear time-dependent absorption. *Journal of Mathematical Sciences*, 2023, Vol. 277, pp. 355–365.
- [4] Aripov M., Matyakubov A., Bobokandov M. Cauchy problem for the heat dissipation equation in a non-homogeneous medium. *AIP Conference Proceedings*, 2023, Vol. 2781, No. 1, 020027.

On the Numerical Stability of Branched Continued Fraction Expansions of Some Ratios of Triple Hypergeometric Functions F_K

Vitaliy Goran

Vasyl Stefanyk Carpathian National University, Ivano-Frankivsk, Ukraine
(e-mail: vitaliy.goran.23@pnu.edu.ua)

The Lauricella–Saran hypergeometric function $F_K(a_1, a_2, b_1, b_2; c_1, c_2, c_3; \mathbf{z})$ is defined as a triple power series that serves as a critical analytical tool in mathematical physics and applied sciences. While these functions are conventionally represented by power series, such representations typically possess restricted convergence domains. Branched continued fractions (BCF) provide a more robust alternative for approximation and analytic continuation. The Backward Recurrence (BR) algorithm is employed to compute the n -th approximant f_n of the BCF expansion for F_K ratios [1]. The algorithm involves the sequential computation of the quantities $Q_k^{(n)}(\mathbf{z})$ for $k = n-1, n-2, \dots, 0$, starting from the initial condition $Q_n^{(n)}(\mathbf{z}) = 1$.

$$Q_k^{(n)}(\mathbf{z}) = 1 - z_{j(k+1)} - \frac{d_{k+1}z_3}{Q_{k+1}^{(n)}(\mathbf{z})}, \quad (1)$$

where $z_{j(k+1)}$ denotes the spatial variables. The index sequence $j(k)$, which determines the alternating subtraction of z_1 and z_2 , is defined by:

$$j(k) = \frac{1 + (-1)^{k+1}}{2} + 1. \quad (2)$$

The BCF coefficients d_k are derived from the hypergeometric parameters a_2, b_1, c_3 according to:

$$d_{2k-1} = \frac{(a_2 + k - 1)(c_3 + k - 1 - b_1)}{(c_3 + 2k - 2)(c_3 + 2k - 1)}, \quad d_{2k} = \frac{(b_1 + k)(c_3 + k - a_2)}{(c_3 + 2k - 1)(c_3 + 2k)}, \quad k \in \mathbb{N}. \quad (3)$$

The n -th approximant is obtained as $f_n = Q_0^{(n)}(\mathbf{z})$.

Theorem 1. Let the relative errors of the elements be bounded by a constant $\alpha \in (0, 1)$. If the parameters of the hypergeometric function satisfy

the constraints $a_2 \geq 0, b_1 \geq -1, c_3 > 0, c_3 \geq b_1$, and $c_3 + 1 \geq a_2$, then $\Omega_L = \{\mathbf{z} \in \mathbb{R}^3 : z_1 \leq 0, z_2 \leq 0, -L \leq z_3 \leq 0\}$ is a set of numerical stability for the BCF expansion (3) for any $L \geq 0$. The relative error $|\epsilon_n^{(f)}|$ is bounded by:

$$|\epsilon_n^{(f)}| \leq \frac{\alpha}{1-\alpha} \left(1 + dL \left(2 + \frac{\alpha}{1-\alpha} \right) \right) \frac{1-\eta^n}{1-\eta}, \quad (4)$$

where $\eta = dL/(1+dL)$ and $d = \sup_{k \in \mathbb{N}} \{|d_k|, |\hat{d}_k|\}$.

Theorem 2. *The set of closed tridisks*

$$\Omega_{d,l} = \left\{ \mathbf{z} \in \mathbb{C}^3 : |z_1| \leq \frac{1-l}{2}, |z_2| \leq \frac{1-l}{2}, |z_3| \leq \frac{l(1-l)}{2d} \right\}, \quad (5)$$

where $l \in (0, 1/3) \cup (1/3, 1)$, is a set of numerical stability for the complex BCF expansion (3). The relative error estimate is given by:

$$|\epsilon_n^{(f)}| \leq \frac{2\alpha(1-l)}{(1-\alpha)(1+l+|1-3l|)} \left(1 + \frac{4l}{1+l+|1-3l|} \left(2 + \frac{\alpha}{1-\alpha} \right) \right) \frac{1-\tau^n}{1-\tau}, \quad (6)$$

where the parameter τ is defined as:

$$\tau = \frac{4l(1-l)}{(1+l)(1+l+|1-3l|) - 4l(1-l)}. \quad (7)$$

In this research, recurrence formulas for the relative errors of the BR-algorithm for approximants of branched continued fraction expansions of Lauricella–Saran hypergeometric function F_K ratios were obtained. Using these formulas, estimates of the relative errors were obtained, and sets of computational stability were constructed in both real and complex domains. Numerical experiments have shown that the BR-algorithm ensures the stability of computations even for high orders of approximants.

References

- [1] R. Dmytryshyn, V. Goran. *Numerical stability of branched continued fraction expansions of Lauricella–Saran’s hypergeometric function F_K ratios*, Demonstr. Math., 2026, 59(1), 20250224.

Neuro-Metaheuristic Approache for Adaptive Vehicle Routing in Urban Transport Networks

Allaoua Hemmak

Mohamed Boudiaf University, M'Sila, Algeria
(e-mail: allaoua.hemmak@univ-msila.dz)

This dissertation addresses the challenge of optimizing vehicle routes in dynamic urban environments, where traffic conditions, customer requests, and vehicle states change during operations. It proposes a hybrid framework, NMFAVRK, that combines Graph Neural Networks (GNN) and a Genetic Algorithm (GA) to solve the Dynamic Vehicle Routing Problem with Time Windows (DVRPTW). The framework builds spatiotemporal customer representations to extract intelligent visit priorities used to initialize and improve the search, then reinforces them with local search and fast route repair when unexpected events such as congestion, breakdowns, or new requests occur. Experiments on the Solomon C102 benchmark show that the system achieved a total distance of 828.9 with a very small gap from the best-known solution and outperformed a random GA baseline by a notable margin. It also responded to disruptions within fractions of a second while reducing fuel consumption under the green routing model. The results confirm that combining deep learning with metaheuristic search improves solution quality and operational flexibility.

Dry Versus Wet Season Dynamics of Cystic Echinococcosis: Insights from Stochastic Extinction Analysis and Deterministic Scenario Comparison

Ibtissam Sannaky, Imane El Berrai, Khalid Adnaoui

Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'Scik, Hassan II University of Casablanca, P.O Box 7955
Sidi Othman, Casablanca, Morocco
(e-mail: bissamsannaky@gmail.com)

Cystic echinococcosis (CE) is a neglected zoonotic disease involving a complex transmission cycle between definitive hosts (dogs), intermediate hosts

(livestock), and humans as accidental dead-end hosts [1]. In this work, we investigate the transmission dynamics of CE through a combined stochastic and deterministic approach. First, we analyze the stochastic behavior of the system using continuous-time Markov chains (CTMC) and multitype branching processes, deriving the probabilities of disease extinction and explosion within host populations. Building on this stochastic framework, we then examine our deterministic compartmental model [2] under two distinct parameter scenarios, corresponding to dry and wet season conditions, in order to assess how seasonal variation in key epidemiological parameters affects disease transmission and persistence. Comparing the outputs of both scenarios reveals significant differences in transmission intensity and the relative contribution of environmental conditions to the disease's spread among hosts. Our findings confirm the role of humans as a dead-end host in the transmission network, while highlighting the importance of seasonal climatic conditions in shaping intervention strategies. This dual approach provides a more comprehensive understanding of CE dynamics and offers practical guidance for the timing and design of control measures in endemic regions such as Morocco.

References

- [1] Cardona et Carmena, A Review of the Global Prevalence, Molecular Epidemiology and Economics of Cystic Echinococcosis in Production Animals, *Veterinary Parasitology* 192 (2013) 10–32.
- [2] Sannaky, I., Riouali, M., Ouldkhouia, N., El berrai, I. and Adnaoui, K., Mathematical modeling of Echinococcosis in humans, dogs and livestock with optimal control strategies. *Iran. J. Numer. Anal. Optim.*, 2025; 15(3): 1075–1115. <https://doi.org/10.22067/ijnao.2025.90038.1530>

Convergence Analysis of $Q4$ and $Q8$ Elements for Elliptical Cavities in Composite Plates

Akhmat Ikramov, Askhad Polatov,
Farrukhkhon Polatov, Wei Guo

*Department of Software Engineering and Artificial Intelligence,
National University of Uzbekistan, Tashkent, Uzbekistan
(e-mail: ikramovaxmat@gmail.com; asad3@yandex.com;
polatov@gmail.com, 709333319@qq.com)*

Fiber-reinforced composites are widely employed in modern engineering because of their excellent strength-to-weight and stiffness-to-weight characteristics. The presence of holes, cavities, and other geometric discontinuities often results in localized stress concentrations that require accurate numerical prediction. While the convergence properties of quadrilateral finite elements are well documented [1], their accuracy in analyzing elliptical cavities in transversely isotropic composites has been studied to a lesser extent [2,3]. To validate the numerical formulation, a benchmark cantilever plate subjected to a uniformly distributed load was first analyzed, and the finite element results were compared with the corresponding analytical solution. The numerical results obtained for successive mesh refinements are presented in Table 1.

Table 1: Numerical results of the benchmark problem.

2*Mesh	2*No.of elements	Linear element ($Q4$)		Quadratic element ($Q8$)	
		Displacement	Stress	Displacement	Stress
		$v \left(\frac{p}{Eh} \right)$	$\sigma_x \left(\frac{p}{h} \right)$	$v \left(\frac{p}{Eh} \right)$	$\sigma_x \left(\frac{p}{h} \right)$
2×1	2	152.533	6.667	214.994	11.833
2×2	4	156.155	6.834	221.119	12.449
3×2	6	184.360	8.938	223.092	12.678
4×3	12	200.797	10.340	224.813	13.396
6×4	24	213.086	11.699	224.974	14.176
8×5	40	218.304	12.476	—	14.613
24×12	288	224.956	15.094	—	—
Exact solution [4]	—	225.000	15.100	225.000	15.100

The results presented in Table 1 demonstrate that both formulations converge toward the analytical solution as the mesh is refined. However, $Q8$

elements achieve engineering accuracy on substantially coarser meshes. The benchmark confirms the correctness of the finite element implementation and illustrates the superior convergence characteristics of quadratic interpolation.

In order to investigate the influence of mesh refinement on the accuracy and convergence of the finite element solution, a numerical experiment was conducted. A rectangular plate, clamped along its left edge and subjected to a uniformly distributed load of $P = -1$ MPa on its right edge, was analyzed. The geometrical and material properties of the plate are as follows: $L = 12$ m, $H = 6$ m, $t = 0.1$ m, $E = 2 \times 10^5$ MPa, $\mu = 0.3$. Here, L and H denote the plate length and height, respectively, t is the plate thickness, E is Young's modulus, and μ is Poisson's ratio.

A rectangular boron–aluminum composite plate containing a centrally located elliptical cavity is considered. The cavity geometry is described by $x^2/a^2 + y^2/b^2 = 1$, where a and b denote the semi-major and semi-minor axes of the ellipse. The cavity aspect ratio is defined as $\lambda = a/b$. The following values were investigated: $\lambda = 1, 2, 4$, and 6 . The reinforcing boron fibers are oriented along the x-axis. The elastic properties of the transversely isotropic composite are $E_1 = 220$ GPa, $E_2 = 27.5$ GPa, $G_{12} = 28.5$ GPa, $\nu_{12} = 0.23$. A uniform tensile stress is applied to the plate boundaries. To accurately capture stress gradients near the cavity boundary, locally refined quadrilateral meshes were employed. The cavity perimeter was discretized using 64 finite elements. The stress concentration factor was evaluated as $K_t = \sigma_{\max}/\sigma_0$, where $\sigma_{\max}$ is the maximum stress at the cavity boundary and σ_0 is the applied nominal stress. The calculated stress concentration factors and relative errors for different cavity geometries are summarized in Table 2. The numerical results

Table 2: Effect of cavity aspect ratio on stress concentration

$\lambda = a/b$	Reference K_t	Q4 K_t	Q8 K_t	Q4 Error (%)	Q8 Error (%)
1	3.00	2.71	2.93	9.67	2.33
2	5.00	4.62	4.95	7.60	1.00
4	9.00	8.01	8.84	11.00	1.78
6	13.00	11.32	12.76	12.92	1.85

reported in Table 2 indicate a substantial increase in stress concentration with increasing cavity aspect ratio [2]. As the cavity becomes more elongated, stress gradients near the cavity tips become increasingly localized, resulting

in larger approximation errors for low-order finite elements. The accuracy of the $Q4$ element deteriorates progressively as λ increases. In contrast, the $Q8$ element maintains stable accuracy throughout the investigated range of cavity geometries. For the most elongated cavity ($\lambda = 6$), the relative error of $Q4$ reaches approximately 13%, whereas the $Q8$ error remains below 2%. These results demonstrate that higher-order interpolation becomes increasingly advantageous as the severity of the stress concentration increases. A comparative study of $Q4$ and $Q8$ quadrilateral finite elements was carried out for stress-concentration problems in transversely isotropic composites. The results show that $Q8$ elements converge considerably faster than $Q4$ elements, while the benchmark analysis confirms the accuracy of the developed finite element formulation. The stress concentration factor increases from 3 to 13 as the cavity aspect ratio rises from $\lambda=1$ to $\lambda=6$. At the same time, the relative error of $Q4$ elements grows to approximately 13%, whereas the $Q8$ error remains below 2.5% for all considered cavity geometries. The influence of elliptical cavity aspect ratio on the accuracy of stress-concentration prediction in a transversely isotropic boron–aluminum composite was quantitatively assessed. The results indicate that quadratic quadrilateral elements provide more accurate and reliable solutions for composite structures containing elongated cavities and other geometric stress concentrators.

References

- [1] O.C. Zienkiewicz, R.L. Taylor, *The Finite Element Method: Its Basis and Fundamentals*, Elsevier, 2005, 752 p.
- [2] A.M. Polatov, A.M. Ikramov, *Numerical Simulation of Nonlinear Strains of Construction Elements*, Advances in Computational Design, 2024, pp. 137–150. DOI: doi.org/10.12989/acd.2024.9.2.137
- [3] S. Natarajan, M. Ganesh, *High-Order Finite Element Approximations for Elasticity Problems Involving Stress Concentrations*, Finite Elements in Analysis and Design, v. 215, 2023. DOI: doi.org/10.1016/j.finel.2023.103915
- [4] A.A. Lukashevich, *Modern Numerical Methods of Structural Mechanics*, Khabarovsk State Technical University Publishing House, Khabarovsk, 2003, 135p.

Applications of Functional Analysis in Physics

Arzu Mammadova^{a,b}, Konul Aliyeva^c

^a*Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan*

^b*Baku Business University, Baku, Azerbaijan*

^c*Sumgayit State University, Sumgayit, Azerbaijan*

(e-mail: arzu.mammad@unec.edu.az, konul.aliyeva@sdu.edu.az)

Functional analysis constitutes the mathematical foundation of modern quantum mechanics. At the beginning of the twentieth century, the development of quantum theory created the necessity to investigate operators acting on infinite-dimensional spaces, which, in turn, stimulated the rapid development of functional analysis. In quantum mechanics, the states of physical systems, observable quantities, and their time evolution are described within the framework of functional analysis.

Mathematical Foundations of Quantum Mechanics. Hilbert Spaces and Quantum States.

In quantum mechanics, the state of a physical system is represented by a **wave function**, which is an element of a complex Hilbert space. A Hilbert space is a complete inner-product space.

Let H be a Hilbert space. Then every quantum state is represented $\psi \in H$ by a vector and satisfies the normalization condition

$$\|\psi\| = 1.$$

The Hilbert space most frequently used in quantum mechanics is $L^2(\mathbb{R}^n)$, which consists of all square-integrable functions.

The inner product is defined by

$$L^2(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty \right\}.$$

and the corresponding norm is given by

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \widehat{g(x)} dx.$$

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

Physical observables and operators. In quantum mechanics, every physical observable is associated with a linear operator.

For example,

1. the **position operator** $(Q\psi)(x) = x\psi(x),;$
2. the **momentum operator** $(P\psi)(x) = -i\hbar \frac{d\psi}{dx},;$
3. the **energy operator (Hamiltonian)** $H = T + V.$

Here,

$$T = -\frac{\hbar^2}{2m}\Delta$$

denotes the kinetic energy operator.

One of the fundamental properties of these operators is **self-adjointness**.

If $A = A^*$ then the spectrum of the operator consists entirely of real numbers, thereby guaranteeing that physical measurements yield real-valued results.

Dirac notation

Dirac's bra-ket notation, which is closely connected with functional analysis, is widely employed in quantum mechanics.

The state vector is written as $\langle|\psi\rangle$

The inner product is denoted by $\langle\phi|\psi\rangle$

The normalization condition is expressed as $\langle\psi|\psi\rangle = 1.$

Time evolution of quantum systems

The time evolution of a quantum state is governed by the **Schrödinger equation**

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

whose solution is written in the form $\psi(t) = e^{-itH/\hbar}\psi_0$

where $U(t) = e^{-itH/\hbar}$ is a **unitary operator group**.

The theory of semigroups and operator exponentials in functional analysis provides the theoretical foundation for the existence and uniqueness of these solutions.

Spectral methods

Spectral methods are among the most powerful tools of functional analysis in physics and applied mathematics.

Spectral decomposition theorem

According to the spectral theorem, for a self-adjoint operator $A = \int_{\sigma(A)} \lambda dE(\lambda)$ where $E(\lambda)$ is the **spectral measure**.

This theorem represents the infinite-dimensional generalization of matrix diagonalization.

Fourier method If $A\phi_n = \lambda_n\phi_n$ then the solution can be represented in the form

$$u = \sum_{n=1}^{\infty} c_n \phi_n$$

This approach is widely applied to the study of: heat conduction, wave propagation, quantum mechanics, acoustics, electromagnetic fields.

Advantages of spectral methods Spectral methods

1. provide high computational accuracy;
2. combine analytical and numerical techniques;
3. enable the solution of complex boundary-value problems;
4. facilitate the determination of the energy levels of quantum systems;
5. are extensively applied in numerous branches of engineering and physics.

Modern applications

Spectral methods are widely employed in the following fields: quantum field theory, nuclear physics, plasma physics, optics, astrophysics, nanoscale systems, photonic crystals, quantum computing.

References

- [1] Reed, M., Simon, B. Methods of Modern Mathematical Physics, Vol. I: Functional Analysis. Academic Press, New York, 1980.
- [2] Hall, B. C. Quantum Theory for Mathematicians. Springer, New York, 2013.
- [3] Davies, E. B. Linear Operators and Their Spectra. Cambridge University Press, Cambridge, 2007.
- [4] Evans, L. C. Partial Differential Equations. American Mathematical Society, Providence, RI, 2010.

Asymptotic Behavior of Self-Similar Solutions for Degenerate Diffusion-Absorption Systems with Time-Dependent Coefficients and Variable Density

Alisher Matyakubov, Abrorjon Mamatov

National University of Uzbekistan, Tashkent, Uzbekistan

(e-mail: a.matyakubov@nuu.uz, mmtovabrорjon1995@gmail.com)

In this paper, we investigate the Cauchy problem in the domain $Q = \{(t, x) : t > 0, x \in \mathbb{R}_+\}$ for a system of nonlinear degenerate parabolic equations with exponentially time-dependent coupling absorption terms:

$$\begin{cases} x^{-n_1} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(x^{-n_2} u^{m_1-1} \frac{\partial u}{\partial x} \right) - x^{-n_3} e^{\alpha_1 t} v^{k_1}, \\ x^{-n_1} \frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left(x^{-n_2} v^{m_2-1} \frac{\partial v}{\partial x} \right) - x^{-n_3} e^{\alpha_2 t} u^{k_2}, \end{cases} \quad (1)$$

$$u(0, x) = u_0(x) \geq 0, v(0, x) = v_0(x) \geq 0, x \in \mathbb{R}_+, \quad (2)$$

where $m_i > 1$, $k_i > 1$, $\alpha_i > 0$ ($i = 1, 2$) are given numerical parameters. The functions $u = u(t, x) \geq 0$ and $v = v(t, x) \geq 0$ are the unknown solutions of the system. The coefficients x^{-n_i} , $n_i \geq 0$, $i = 1, 2, 3$, describe the influence of the inhomogeneous medium. The problem (1)-(2) can be interpreted as a mathematical model describing diffusion and absorption processes of two interacting substances, species, or quantities evolving in a nonlinear medium [1, 2]. The system incorporates both self-diffusion effects and mutual interactions between the components, which are governed by nonlinear and time-dependent mechanisms [1, 2, 5]. A common and effective approach for studying solutions of such problems is based on self-similar analysis.

Below we provide a method of nonlinear splitting [1, 3, 4] to construct self-similar equation. For construction of the self-similar solutions of the system (1) $u(t, x), v(t, x)$ in the form:

$$u(t, x) = \bar{u}(t)w(x, \tau(t)), \quad v(t, x) = \bar{v}(t)\varphi(x, \tau(t)). \quad (3)$$

The time-dependent factors are chosen as

$$\bar{u}(t) = A_1 e^{a_1 t}, \quad \bar{v}(t) = A_2 e^{a_2 t},$$

where

$$A_i = (a_i a_{3-i}^{k_i})^{\frac{1}{k_i k_{3-i} - 1}}, a_i = \frac{k_i \alpha_{3-i} - \alpha_i}{k_i k_{3-i} - 1}, k_i \alpha_{3-i} > \alpha_i, k_i k_{3-i} > 1, i = 1, 2.$$

To ensure the invariance of system (1) under the self-similar transformation (3), the corresponding scaling exponents must satisfy a compatibility condition linking the nonlinear diffusion terms of both equations. In particular, the requirement that the diffusion operators retain the same homogeneity under the similarity scaling leads to the relation $(\alpha_1 + k_1 \alpha_2)(m_1 - 1) = (\alpha_2 + k_2 \alpha_1)(m_2 - 1)$. Furthermore, balancing the amplitudes of the rescaled variables gives rise to the proportionality coefficient $l = \frac{A_2^{m_1-1}}{A_1^{m_2-1}}$, which connects the similarity constants A_1 and A_2 . Using these compatibility conditions, one derives the parameters $d_1 = \frac{1}{(m_1-1)}$, $d_2 = \frac{l}{(m_2-1)}$, which appear in the rescaled time-dependent correction terms of the self-similar formulation [5]. Substituting the similarity representation (1) into system (3) then yields the transformed system

$$\begin{cases} \frac{\partial w}{\partial \tau} = x^{n_1} \frac{\partial}{\partial x} (x^{-n_2} w^{m_1-1} \frac{\partial w}{\partial x}) - d_1 \tau^{-1} (w + x^\mu \varphi^{k_1}), \\ \frac{\partial \varphi}{\partial \tau} = x^{n_1} \frac{\partial}{\partial x} (x^{-n_2} \varphi^{m_2-1} \frac{\partial \varphi}{\partial x}) - l d_2 \tau^{-1} (\varphi + x^\mu w^{k_2}), \end{cases} \quad (4)$$

where

$$\tau(t) = \begin{cases} A_i^{m_i-1} \frac{e^{(\alpha_i + k_i \alpha_{3-i})(m_i-1)t}}{(\alpha_i + k_i \alpha_{3-i})(m_i-1)}, & \text{at } (\alpha_i + k_i \alpha_{3-i})(m_i-1) > 0, \end{cases}$$

$$\mu = n_1 - n_3, \quad i = 1, 2.$$

We seek the self-similar solution of system (4) in the following form:

$$w(\tau, x) = f(\xi), \quad \varphi(\tau, x) = \phi(\xi), \quad \xi = \frac{x}{\tau^{\frac{1}{\sigma}}}, \quad \sigma = 2 + n_2 - n_1. \quad (5)$$

Substituting (4) into (5), we obtain the following self-similar system:

$$\begin{cases} \xi^{n_1} \frac{d}{d\xi} \left(\xi^{-n_2} f^{m_1-1} \frac{df}{d\xi} \right) + \frac{\xi}{\sigma} \frac{df}{d\xi} - d_1 (f + \xi^\mu \phi^{k_1}) = 0, \\ \xi^{n_1} \frac{d}{d\xi} \left(\xi^{-n_2} \phi^{m_2-1} \frac{d\phi}{d\xi} \right) + \frac{\xi}{\sigma} \frac{d\phi}{d\xi} - l d_2 (\phi + \xi^\mu f^{k_2}) = 0. \end{cases} \quad (6)$$

Suppose that for system of equations (6) the following conditions are fulfilled:

$$f'(0) = 0, \quad \phi'(0) = 0. \quad (7)$$

Theorem 1. *Let $h_1 > 0$ and $h_2 > 0$. Then the compactly supported solution of problem (6), (7) has the following asymptotic behavior as $\xi \rightarrow a^{1/\sigma}$:*

$$f(\xi) = B_1 \Theta_1(a - \xi^\sigma)^{c_1}(1 + o(1)), \phi(\xi) = B_2 \Theta_2(a - \xi^\sigma)^{c_2}(1 + o(1)), \quad (8)$$

where the coefficients $c_i = \frac{1}{m_i-1}$, $h_i = k_i c_{3-i} - c_i + 1$, $\Theta_i = \left(\frac{1}{c_i B_i^{m_i-1} \sigma^2}\right)^{c_i}$, $B_i > 0$, $i = 1, 2$. Thus, the theorem characterizes the principal asymptotic profile of the solution near the free boundary. The results obtained for problem (7)-(8) were applied within the framework of the research project funded by the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan under the research grant AL-9224104601, titled “Mathematical modeling of processes described by nonlinear, divergent, and non-divergent parabolic equations and systems”.

References

- [1] M. Aripov, M. Bobokandov, *Blow-up analysis for a doubly nonlinear parabolic non-divergence form equation with source term*, Bulletin of the Institute of Mathematics, 2024, 5(4), 7-21.
- [2] A. Matyakubov, D. Raupov, *Explicit estimate for blow-up solutions of non-linear parabolic systems of non-divergence form with variable density*, AIP Conference Proceedings, 2023, 2781, 020055.
- [3] A. Matyakubov, D. Raupov, *On some properties of the blow-up solutions of a nonlinear parabolic system*, Lecture Notes in Civil Engineering, 2022, 180, 289-301.
- [4] A. Mamatov, M. Mamatkulova, *Blow-up and global analysis of the destruction of complex diffusion processes*, AIP Conference Proceedings, 2025, 3356, 040008.
- [5] M. Bedjaoui, Ph. Souplet, *Critical blow-up exponents for a system of semi-linear parabolic equations with absorption*, Proceedings of the Royal Society of Edinburgh A, 2000, 130, 1-17.

On the Asymptotic Behaviour of Solutions to a Nonlinear Parabolic System in Non-Divergence Form with Variable Density

Alisher Matyakubov^a, Jasurbek Salimov^{a,b}

^aNational University of Uzbekistan, Tashkent, Uzbekistan

^bInternational Digital University, Tashkent, Uzbekistan
(e-mail: salimov-j@nuu.uz)

In this work, in the domain $Q = \{(t, x) : t > 0, x \in \mathbb{R}\}$ we consider the Cauchy problem for a system of nonlinear parabolic equations not in divergence form:

$$\begin{cases} |x|^\ell \frac{\partial u}{\partial t} = u^{\alpha_1} \frac{\partial}{\partial x} \left(u^{m-1} \frac{\partial u}{\partial x} \right) - |x|^\ell v^{\beta_1}, \\ |x|^\ell \frac{\partial v}{\partial t} = v^{\alpha_2} \frac{\partial}{\partial x} \left(v^{m-1} \frac{\partial v}{\partial x} \right) - |x|^\ell u^{\beta_2}. \end{cases} \quad (1)$$

$$\begin{cases} u(0, x) = u_0(x) \geq 0, \\ v(0, x) = v_0(x) \geq 0. \end{cases} \quad (2)$$

where $\beta_1 > 1, \beta_2 > 1, m > 1, \alpha_1 > 1, \alpha_2 > 1, \ell > 0$ are numerical parameters, $u = u(t, x) \geq 0, v = v(t, x) \geq 0$ are the solutions.

In this paper, we study a nonlinear parabolic system of equations in non-divergence form with variable density. Such systems naturally arise in the mathematical modeling of combustion processes and heat transfer. The qualitative properties of solutions to nonlinear parabolic systems of this type have been investigated by various authors [1, 2].

Here, we introduce the notations:

$$\mu_{i1} = \frac{1 - \alpha_i}{(\alpha_i + m - 1)^2}, \quad \mu_{i2} = \frac{B_i^{1-\alpha_i-m}}{(\ell + 2)^2(\alpha_i + m - 1)}, \quad \mu_{i3} = -\frac{d_i B_{3-i}^{\beta_i}}{(\ell + 2)^2 B_i^{\alpha_i+m}},$$

$$\zeta_i = \frac{\beta_i}{\alpha_{3-i} + m - 1} - \frac{1}{\alpha_i + m - 1} + 1, \quad \gamma = n_1(\alpha_1 + m - 1) + 1, \quad b_i = A_i B_i \quad (i = 1, 2).$$

Let $y(\eta) = y^0 + o(1), \quad z(\eta) = z^0 + o(1) \quad$ as $\eta \rightarrow +\infty$ and the equality $n_2(\alpha_2 + m - 1) = n_1(\alpha_1 + m - 1)$ is performed.

Theorem 1. Let $\zeta_1 = 0, \zeta_2 > 0$. The compactly supported solution to problem (1)-(2) has the following asymptotic behavior as $x \rightarrow a^{\frac{1}{2}}(t - T)^{\frac{\gamma}{2}}$:

$$\begin{aligned} u_A(t, x) &\simeq b_1(T - t)^{-n_1} \left(a - \frac{\gamma x^{\ell+2}}{A_1^{\alpha_1+m-1}(T - t)^\gamma} \right)^{\frac{1}{\alpha_1+m-1}} (y^0 + o(1)), \\ v_A(t, x) &\simeq b_2(T - t)^{-n_2} \left(a - \frac{\gamma x^{\ell+2}}{A_2^{\alpha_2+m-1}(T - t)^\gamma} \right)^{\frac{1}{\alpha_2+m-1}} (z^0 + o(1)). \end{aligned} \quad (3)$$

where $0 < y^0 < +\infty$ and $0 < z^0 < +\infty$ are the solutions of the following nonlinear algebraic system:

$$\begin{aligned} \mu_{11}\omega_1^{\alpha_1+m} + \mu_{12}\omega_1 &= 0, \\ \mu_{21}\omega_2^{\alpha_2+m} + \mu_{22}\omega_2 + \mu_{23}\omega_1^{\beta_2} &= 0. \end{aligned} \quad (4)$$

When an exact solution cannot be obtained explicitly, deriving estimates or determining the asymptotic behaviour of the solution is an important mathematical result. Therefore, this work constructs an asymptotic representation of self-similar solutions for a nonlinear parabolic system in non-divergence form $(-)$. In addition, the system of equations is solved numerically, and the corresponding graphical representations are constructed. The results obtained are used to analyze the qualitative properties of the solutions.

References

- [1] M. Aripov, O. Atabaev, A. Al-Marashi, *On the behavior of solutions of a doubly nonlinear degenerate parabolic system with nonlinear sources and absorptions with variable densities*, Bulletin of the Karaganda University, 2025, No. 1, pp. 12–23.
- [2] A. S. Matyakubov, D. R. Raupov, *Explicit estimate for blow-up solutions of nonlinear parabolic systems of non divergence form with variable density*, AIP Conference Proceedings, 2023, 2781, 020055.

Dynamical Analysis of Hantavirus Infection Model

Mahmoud Moustafa

*College of Engineering and Information Technology, Onaizah Colleges,
Qassim, Saudi Arabia*

(e-mail: mahmoudmoustafa949@gmail.com)

In this study, a Hantavirus infection model incorporating predators of mice is formulated and analyzed. The populations are divided into susceptible mice, infected mice and predators of mice. Mathematical analysis and numerical simulations are performed to clarify the characteristics of the proposed Hantavirus infection model. The existence, uniqueness, non-negativity and boundedness of the solutions are examined. The basic reproduction number is identified. The local and global stability of the equilibrium points of the model is studied. The mathematical proof of the existence of forward bifurcation is given. The theoretical findings are illustrated by numerical simulations. The impact of predators of mice on the stability of the Hantavirus infection model is studied. The results suggest the possibility of control of the spread of the epidemic by introducing predators in a suitable way so that such control does not kill all the mice but lowers the epidemic spread.

Sequential Fixed-Width Confidence Intervals for Right-Censored Data via Averaged Stochastic Approximation

Nargiza Nurmukhamedova, Nargiza Boltaeva

National University of Uzbekistan, 4 University St., Tashkent, Uzbekistan.

(e-mail: n.nurmuhammedova@nuu.uz, boltayevan27@gmail.com)

Let $X_1, X_2, \dots$ be i.i.d. lifetimes with density $f(\cdot; \theta_0)$, $\theta_0 \in \mathbb{R}$, censored on the right by independent times C_i , so that one observes $Y_i = \min(X_i, C_i)$ and $\delta_i = \mathbf{1}\{X_i \leq C_i\}$. Let $\psi(y, \delta, \theta)$ be the censored score (the conditional expectation of the full-data score given (Y, δ)), with $\mathbb{E}_{\theta_0} \psi(Y, \delta, \theta_0) = 0$ and Fisher information $I = \mathbb{E}_{\theta_0} [\psi(Y, \delta, \theta_0)^2] > 0$. We address the construction of a

fixed-width confidence interval of half-width ε and level γ for θ_0 from a data stream, when I depends on the unknown censoring law and must be estimated online.

We run the projected Robbins–Monro recursion with gain $\gamma_n = c/n$ and its Polyak–Ruppert average,

$$\theta_{n+1} = \Pi_K \left[\theta_n + \frac{c}{n} \psi(Y_{n+1}, \delta_{n+1}, \theta_n) \right], \quad \bar{\theta}_n = \frac{1}{n} \sum_{k=1}^n \theta_k, \quad (1)$$

together with the recursive information estimate $\hat{I}_n = n^{-1} \sum_{i=1}^n \psi(Y_i, \delta_i, \bar{\theta}_{i-1})^2$. With $a = \Phi^{-1}((1 + \gamma)/2)$ and a floor $\iota \in (0, I)$, define the stopping time and the interval

$$N_\varepsilon = \inf \left\{ n \geq n_0 : n \geq \frac{a^2}{(\hat{I}_n \vee \iota) \varepsilon^2} \right\}, \quad I(N_\varepsilon) = [\bar{\theta}_{N_\varepsilon} - \varepsilon, \bar{\theta}_{N_\varepsilon} + \varepsilon]. \quad (2)$$

The controlling scale is the asymptotic *variance* I^{-1} , not the information: the Chow–Robbins optimal nonrandom size is $n_\varepsilon = a^2 I^{-1} / \varepsilon^2$, so a rule using $\hat{I}_n$ in place of $\hat{I}_n^{-1}$ stops a factor I^2 too early.

The averaged estimator is asymptotically linear with the efficient influence function $\tilde{\psi}_{\theta_0} = I^{-1} \psi(\cdot, \theta_0)$, hence asymptotically efficient in the Hájek–Le Cam sense; its limiting variance I^{-1} attains the convolution bound. Our main result is the following.

Theorem 1. *Under standard regularity conditions and the gain condition $cI > \frac{1}{2}$, the procedure (1)–(2) is asymptotically consistent and asymptotically efficient in the Chow–Robbins sense; that is, for every $\gamma \in (0, 1)$,*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\theta_0 \in I(N_\varepsilon)\} = \gamma, \quad \frac{N_\varepsilon}{n_\varepsilon} \rightarrow 1 \text{ a.s. and in mean.} \quad (3)$$

The proof rests on an asymptotically linear (Bahadur) representation $\bar{\theta}_n - \theta_0 = \mathbb{P}_n \tilde{\psi}_{\theta_0} + R_n$ with $\sqrt{n} R_n \rightarrow 0$, whose remainder is controlled almost surely through the martingale structure of the score; the random-index passage uses Anscombe’s theorem, and the stopping-time efficiency follows from the strong consistency of $\hat{I}_n$. A distinctive feature is that stopping-time efficiency $N_\varepsilon/n_\varepsilon \rightarrow 1$ holds for *every* gain $c > 0$, whereas the gain condition $cI > \frac{1}{2}$ governs coverage only. A simulation study on censored exponential models confirms the theory and quantifies the finite-sample coverage as a function of cI .

References

- [1] Y.S. Chow, H. Robbins, *On the asymptotic theory of fixed-width sequential confidence intervals for the mean*, Annals of Mathematical Statistics, 1965, 36, 457–462.
- [2] B.T. Polyak, A.B. Juditsky, *Acceleration of stochastic approximation by averaging*, SIAM Journal on Control and Optimization, 1992, 30(4), 838–855.
- [3] A.W. van der Vaart, *Asymptotic Statistics*, Cambridge University Press, Cambridge, 1998.

Mathematical Modeling of Production Efficiency Under Different Constraints

Aygun Omarova^a, Tofiq Orujov^b

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Mingachevir State University, Mingachevir, Azerbaijan*
(e-mail: orudjova.aygun@gmail.com)

The efficient allocation of limited production resources is one of the principal problems in mathematical economics. In modern enterprises, several products are manufactured simultaneously while sharing common financial, labor, and material resources. Consequently, minimizing the production cost of each product separately does not necessarily lead to the minimum total production cost. Mathematical optimization methods provide an effective framework for solving such problems under various technological and economic constraints.

One of the most powerful techniques for constrained optimization is the method of Lagrange multipliers. This method transforms a constrained optimization problem into an unconstrained one by incorporating the constraints into the objective function. Suppose a firm produces n products using m limited resources. Let $c_i = f_i(q_{1i}, q_{2i}, \dots, q_{mi})$ denote the production cost of the i -th product, where q_{hi} represents the amount of the h -th resource allocated to that product. The optimization problem is to minimize

$$\sum_{i=1}^n c_i$$

subject to the resource constraints

$$\sum_{i=1}^n q_{hi} = Q_h, \quad h = 1, \dots, m.$$

The corresponding Lagrange function is

$$L = \sum_{i=1}^n c_i + \sum_{h=1}^m \lambda_h \left(\sum_{i=1}^n q_{hi} - Q_h \right),$$

where λ_h are the Lagrange multipliers. The necessary optimality conditions are obtained from

$$\frac{\partial L}{\partial q_{hi}} = 0.$$

The resulting multipliers determine the marginal value (shadow price) of each limited resource and characterize the economically optimal distribution of production inputs.

Another important class of optimization problems concerns maximizing production revenue. Assume that two products are produced using the same limited resources and that their production functions are

$$x_1 = F_1(x_3, x_4), \quad x_2 = F_2(x_5, x_6).$$

If p_1 and p_2 denote the corresponding product prices, then the objective is to maximize

$$p_1 x_1 + p_2 x_2$$

subject to technological and resource constraints. Application of the Lagrange multiplier method leads to the conditions

$$p_1 \frac{\partial F_1}{\partial x_3} = p_2 \frac{\partial F_2}{\partial x_5},$$

$$p_1 \frac{\partial F_1}{\partial x_4} = p_2 \frac{\partial F_2}{\partial x_6}.$$

These equalities imply that, at the optimum, the marginal productivity of each common production factor is identical in all production activities.

A more general optimization problem involves maximizing a production function

$$y = F(x_1, x_2, \dots, x_n)$$

under the budget constraint

$$p_1x_1 + p_2x_2 + \dots + p_nx_n = C.$$

The corresponding Lagrange function is

$$\Phi = F(x_1, \dots, x_n) - \lambda(p_1x_1 + \dots + p_nx_n - C).$$

The optimality conditions become

$$\frac{\partial F}{\partial x_i} = \lambda p_i, \quad i = 1, \dots, n.$$

Thus, the marginal productivity of every production factor is proportional to its price, providing an economically justified criterion for efficient resource allocation under limited financial resources.

The considered mathematical models demonstrate that the method of Lagrange multipliers is an effective analytical tool for solving production planning problems. It allows the determination of optimal production strategies, minimizes production costs, maximizes enterprise revenue, and improves the efficiency of resource utilization. The proposed approach can be successfully applied in industrial enterprises, regional economic planning, and decision-support systems.

References

- [1] Zaytsev M.G., Varyukhin S.E. *Methods of Management Optimization and Decision Making*. Moscow, 2008.
- [2] *Mathematical Modeling: Problems and Results*. Moscow, 2003.
- [3] Thomas R. *Quantitative Methods for Business Analysis*. Moscow, 1999.

Geometric Effects on Stress Localization in Fiber Composites

Askhad Polatov, Akhmat Ikramov,
Sukhbatulla Pulatov, Uchkunbek Adambaev

*Department of Software Engineering and Artificial Intelligence,
National University of Uzbekistan, Tashkent, Uzbekistan
(e-mail: asad3@yandex.com; ikramovaxmat@gmail.com;
s.pulatov@nuu.uz; u.adambaev@nuu.uz)*

Fiber composites are widely used in engineering because of their high stiffness and strength to weight ratios. However, holes, cutouts, and service-induced defects act as stress concentrators that can initiate damage. In anisotropic composites, stress redistribution is strongly affected by fiber–matrix interaction, requiring accurate prediction of localized stresses and plastic zones for reliable structural assessment [4].

This study investigates elastoplastic deformation and stress localization in transversely isotropic composites containing elliptical holes and straight cracks. The material is modeled as an equivalent homogeneous continuum with effective transversely isotropic properties. Elastoplastic behavior is described within the small-deformation framework using an iterative procedure based on Ilyushin’s theory [1].

Table 3: Mesh convergence study

Mesh	Type	Elements	Nodes	$\sigma_{\max}$ (MPa)	Error
M1	HEX8	248	387	83.1	5.6%
M2	HEX8	1024	1458	86.7	1.5%
M3	HEX8	4096	5625	88.0	0.2%

The boundary-value problem is solved by the finite element method using eight-node isoparametric hexahedral elements (HEX8) with local mesh refinement near holes and crack tips. The formulation is validated against a benchmark problem of a boron–aluminum composite plate with a central circular hole. Close agreement with published explicit-fiber solutions [2,3] confirms the applicability of the homogenized approach to highly reinforced composites. Consistent with previous finite-element studies of composite structures



Figure 1: Distribution of stress intensity near an elliptical hole ($r_2/r_1 = 1/8$).

containing discontinuities [5], a convergence analysis for a plate with an elliptical hole demonstrates monotonic convergence and mesh-independent results through substantial error reduction under mesh refinement (Table 1).

Parametric simulations were performed to quantify the influence of the ellipse aspect ratio (r_2/r_1) on stress localization. The results show that the opening geometry governs both the magnitude and distribution of stresses. For circular and moderately elongated holes, maximum stress intensities occur near the upper and lower portions of the contour. As the opening becomes flatter, the localization pattern shifts toward the lateral boundaries. Figure 1 illustrates the stress-intensity distribution near an elliptical hole with ($r_2/r_1 = 1/8$). Elevated stress concentrations are localized at the lateral boundaries due to increased contour curvature. These regions represent potential sites of matrix–fiber debonding and damage initiation.

A localization transition is observed for highly flattened openings ($r_2/r_1 < 0.2$), where elevated shear stresses spread along a large portion of the hole boundary. The results indicate that stress localization depends not only on defect size but also on its geometry. As the hole becomes flatter, the region of maximum shear stress expands significantly, increasing the susceptibility of the matrix to damage.

Elastoplastic analyses of plates containing elliptical holes and straight cracks reveal substantial stress redistribution due to matrix yielding. Compared with elastic solutions, plastic deformation reduces peak stress intensities by approximately 20–25% while enlarging the affected region. For elliptical holes, plastic zones are concentrated near the upper and lower portions of the contour, whereas in cracked plates they develop mainly along the crack faces, while the highest stresses remain localized at the crack tips.

The study presents a validated three-dimensional elastoplastic finite-element framework for transversely isotropic fiber composites with geometric stress con-

centrators and, for the first time, quantitatively characterizes the influence of ellipse aspect ratio on shear-stress localization and matrix–fiber debonding in boron–aluminum composites. A localization transition is identified for highly flattened openings ($r_2/r_1 < 0.2$), critical debonding-prone regions are determined through shear-stress analysis, and elastic and elastoplastic stress redistribution mechanisms are systematically compared for structures containing holes and cracks.

The results demonstrate that defect geometry is a governing factor in stress localization. Flattened elliptical holes produce extended regions of elevated shear stress along their lateral boundaries, whereas cracks generate highly concentrated stress fields confined to crack tips. Matrix plasticity causes significant stress redistribution and reduces peak stress intensity by 20–25%, enabling more realistic identification of regions prone to debonding and damage initiation. The proposed methodology is applicable to the design and integrity assessment of composite structures containing holes, cutouts, and crack-like defects.

References

- [1] B.E. Pobedrya, *Models of Continuum Mechanics*, Fundamental and Applied Mathematics, 1997.
- [2] A.M. Polatov, A.M. Ikramov, A.A. Khaldjigitov, *Computer Modeling of Elastoplastic Stress State of Fibrous Composites with Hole*, Coupled Systems Mechanics, 2019, 8(4), pp. 299–313.
- [3] A.M. Polatov, A.M. Ikramov, S.I. Pulatov, S.M. Gaynazarov, *Numerical Modeling of the Stress State of Constructions from Fibrous Composites*, Journal of Physics: Conference Series, 1479, 012100, 2020.
- [4] N.K. Jain, N.D. Mittal, *Finite Element Analysis for Stress Concentration and Deflection in Composite Plates with a Central Circular Hole*, Materials Science and Engineering A, 2008, 498, pp. 115–124.
- [5] Y. Wang, H. Zhang, X. Liu, *Numerical Investigation of Stress Localization in Perforated Fiber-Reinforced Composites*, Composite Structures, Vol. 340, 118247, 2024.

On Numerical Solution to Inverse Problem of Restoring Special Type Source of Parabolic Equation

Anar Rahimov

Baku State University, Z. Khalilov 33, AZ1148, Baku, Azerbaijan

Institute of Mathematics, Baku, Azerbaijan

(e-mail: anar_r@yahoo.com)

We consider the inverse source problem with respect to the parabolic equation:

$$\begin{aligned} \frac{\partial v(x,t)}{\partial t} &= a(x,t) \frac{\partial^2 v(x,t)}{\partial x^2} + a_1(x,t) \frac{\partial v(x,t)}{\partial x} + a_2(x,t) v(x,t) + \\ &+ f(x,t) + \sum_{s=1}^L B_s(x,t) C_s(x), \\ (x,t) \in \Omega &= \{(x,t) : 0 < x < l, 0 < t \leq T\}, \end{aligned} \quad (1)$$

under the following initial-boundary conditions and additional conditions:

$$v(x,0) = \varphi_0(x), \quad x \in [0, l], \quad (2)$$

$$v(0,t) = \psi_0(t), \quad v(l,t) = \psi_1(t), \quad t \in [0, T], \quad (3)$$

$$v(x, \bar{t}_s) = \varphi_{1s}(x), \quad x \in [0, l], \quad \bar{t}_s \in (0, T], \quad s = 1, \dots, L. \quad (4)$$

Here, $L > 0$ is a given integer; $\bar{t}_s \in (0, T]$, $s = 1, \dots, L$ are given instants of time; given functions $a_0(x,t) > 0$, $a_1(x,t)$, $a_2(x,t)$, $f(x,t)$, $B_s(x,t)$, $\varphi_0(x)$, $\psi_0(t)$, $\psi_1(t)$, and $\varphi_{1s}(x)$, $s = 1, \dots, L$ are continuous with respect to x and t ; $B_s(x,t)$ are linearly independent functions differentiable with respect to t , and $a_2(x,t) \leq 0$, $B_s(x,t) \geq 0$, $\frac{\partial B_s(x,t)}{\partial t} \geq 0$. Functions $\varphi_0(x)$, $\varphi_{1s}(x)$, $\psi_0(t)$ and $\psi_1(t)$ satisfy the consistency conditions:

$$\begin{aligned} \varphi_0(0) &= \psi_0(0), \quad \varphi_0(l) = \psi_1(0), \\ \varphi_{1s}(0) &= \psi_0(\bar{t}_s), \quad \varphi_{1s}(l) = \psi_1(\bar{t}_s), \quad s = 1, \dots, L. \end{aligned}$$

The problem (1)-(4) consist in finding the unknown continuous L -dimensional vector function $C(x) = (C_1(x), \dots, C_L(x))^T$ and the respective solution of the boundary value problem $v(x,t)$, which is twice continuously differentiable

with respect to x and once continuously differentiable with respect to t for $(x, t) \in \Omega$, satisfying conditions (1)-(4).

We propose an approach to the numerical solution to problem (1)-(4), based on using the method of lines and the special representation of the solution. The problem (1)-(4) reduces to a system of ordinary differential equations with unknown parameters. To determine these parameters, a method is proposed based on the results by author [1, 2].

In the domain Ω , let us set up the lines $x_i = ih_x$, $i = 0, 1, \dots, N$, $h_x = l/N$. On these lines, we define functions $v_i(t) = v(x_i, t)$, $t \in [0, T]$, $i = 0, 1, \dots, N$, for which according to (2)-(5)

$$v_i(0) = \varphi_0(x_i) = \varphi_{0i}, \quad i = 0, \dots, N, \quad (5)$$

$$v_0(t) = \psi_0(t), \quad v_N(t) = \psi_1(t), \quad t \in [0, T], \quad (6)$$

$$v_i(\bar{t}_s) = \varphi_{1s}(x_i) = \varphi_{1s,i}, \quad \bar{t}_s \in (0, T], \quad s = 1, \dots, L, \quad i = 0, \dots, N. \quad (7)$$

On the lines $x = x_i$, we approximate the derivatives $\partial v / \partial x$, $\partial^2 v / \partial x^2$ with the use of central difference schemes and we use the notations

$$\begin{aligned} a_i(t) &= a(x_i, t), \quad \bar{f}_i(t) = f(x_i, t), \quad a_{1i}(t) = a_1(x_i, t), \quad a_{2i}(t) = a_2(x_i, t), \\ B_{si}(t) &= B_s(x_i, t), \quad C_{si} = C_s(x_i), \quad s = 1, \dots, L, \quad i = 1, \dots, N-1. \end{aligned}$$

Then we obtain the system of ordinary differential equations of $(N-1)$ th order with unknown (identified) vector of parameters $C_s = (C_{s1}, \dots, C_{s,N-1})^T$:

$$\dot{v}(t) = A(t)v(t) + f(t) + \sum_{s=1}^L E_{sB(t)} C_s, \quad t \in (0, T], \quad (8)$$

$$v(0) = \varphi_0, \quad (9)$$

$$v(\bar{t}_s) = \varphi_{1s}, \quad s = 1, \dots, L, \quad (10)$$

where $v(t) = (v_1(t), \dots, v_{N-1}(t))^T$, $B_s(t) = (B_{s1}(t), \dots, B_{s,N-1}(t))^T$, $\varphi_0 = (\varphi_{01}, \dots, \varphi_{0,N-1})^T$, $\varphi_{1s} = (\varphi_{1s,1}, \dots, \varphi_{1s,N-1})^T$, $E_{sB(t)}$ is $(N-1)$ -dimensional square matrix whose i -th element on the principal diagonal is equal to the i -th component of vector $B_s(t)$, i. e. $B_{si}(t)$; all the other elements are zero. The matrix A is a square $(N-1)$ -dimensional three-diagonal matrix with known elements and the vector $f(t)$ is an $(N-1)$ -dimensional vector of known elements.

The following theorem holds. Denote by $0_{(N-1) \times (N-1)}$ zero $(N - 1)$ - dimensional square matrix.

Theorem. *Let $\alpha_s(t)$, $s = 1, \dots, L$, which is a square matrix function and $\gamma(t)$, which is a vector function of dimension $(N - 1)$, be solution of the following Cauchy problems:*

$$\dot{\alpha}_s(t) = A(t) \alpha_s(t) + E_s B(t), \quad \alpha_s(0) = 0_{(N-1) \times (N-1)}, \quad s = 1, \dots, L, \quad (11)$$

$$\dot{\gamma}(t) = A(t) \gamma(t) + f(t), \quad \gamma(0) = \varphi_0. \quad (12)$$

Then for an arbitrary constant $(N - 1)$ -dimensional vector C_s the solution to the Cauchy problem (11), (12) is the following vector-function:

$$v(t) = \sum_{s=1}^L \alpha_s(t) C_s + \gamma(t). \quad (13)$$

To determine $\alpha_s(t)$, $s = 1, \dots, L$, we solve independently the matrix Cauchy problem (11), and to find vector-function $\gamma(t)$, we solve the Cauchy problem (12), and by using condition (10) and representation (13), we obtain the equality

$$v(\bar{t}_s) = \varphi_{1s} = \sum_{s=1}^L \alpha_s(\bar{t}_s) C_s + \gamma(\bar{t}_s), \quad s = 1, \dots, L, \quad (14)$$

which is a system of algebraic equations of $(N - 1)$ th-order from which one can find the required vector C_s , $s = 1, \dots, L$.

The results of numerical experiments on test problems will be presented in the report.

References

- [1] K.R. Aida-zade, A.B. Rahimov *An approach to numerical solution of some inverse problems for parabolic equations*, Inverse Problems in Science and Engineering, 2014, 22(1), 96-111.
- [2] K.R. Aida-zade, A.B. Rahimov *On recovering space or time-dependent source functions for a parabolic equation with nonlocal conditions*, Applied Mathematics and Computation, 2022, 419, 126849 (17 pages)

On the Properties of Solutions to a Multidimensional Nonlinear Heat Conduction Problem with Coupled Nonlinear Boundary Conditions

Zafar Rakhmonov^a, Aziza Zaripova^b

^aNational University of Uzbekistan, Tashkent, Uzbekistan

^bKarshi State University, Karshi, Uzbekistan

(e-mail: zraxmonov@inbox.ru, azizazaripova1990@mail.ru)

In the present work, we study the following problem posed in the domain:
 $Q_+ = \{(x, t) : x \in R_+^N, 0 < t < T\}$

$$\begin{cases} u_{it} = \nabla (|\nabla u_i^{m_i}|^{p_i} \nabla u_i^{m_i}), & x \in R_+^N, 0 < t < T, \\ -|\nabla u_i^{m_i}|^{p_i} \frac{\partial u_i^{m_i}}{\partial x_1} = u_{i+1}^{q_i}, & u_{k+1} = u_1, x_1 = 0, 0 < t < T, \\ u_i(x, 0) = u_{i0}(x), & x \in R_+^N, \end{cases} \quad (1)$$

here, m_i and p_i denote numerical parameters characterizing the nonlinear properties of the medium and are assumed to satisfy the following condition: $p_i > \frac{1}{m_i} - 1$, $q_i > 0$, $u_{i0}(x)$ —continuous, nonnegative, and bounded functions, where $-|\nabla u_{i0}^{m_i}|^{p_i} \frac{\partial u_{i0}^{m_i}}{\partial x_1} = u_{(i+1)0}^{q_i}$, $x_1 = 0$, and for some. Since the system of equations (1) is degenerate, it may not possess a classical solution whenever $u_i(x, t) = 0$ or $|\nabla u_i^{m_i}| = 0$. Therefore, we consider a weak solution of problem (1) in the class $|\nabla u_i^{m_i}|^{p_i} \nabla u_i^{m_i} \in C(R_+^N \times (0, +\infty))$, $0 < u_i^{m_i} \in C(Q)$, ($i = 1, 2, \dots, k$) which satisfies the system of equations (1) in the sense of the corresponding integral identity, or equivalently, in the sense of distributions [1]-[2].

Problem (1) describes a wide range of physical and biological processes occurring in multicomponent nonlinear media encountered in the natural sciences. In particular, for different values of the numerical parameters, it models cross-diffusion processes, heat transfer between different bodies, combustion phenomena, the polytropic filtration of liquids and gases, as well as population dynamics of biological species and various other processes [3]-[6].

Theorem 1. Assume that the condition $\min \{N\beta_i - \alpha_i\} > 0$ holds. Then the solution to problem (1) exists globally, where

$$\alpha_i + 1 = \beta_i + (\alpha_i m_i + \beta_i)(p_i + 1), \quad (\alpha_i m_i + \beta_i)(p_i + 1) = \alpha_{i+1} q_i.$$

In this work, we investigated the conditions for the global-in-time existence of solutions to a nonlinear heat propagation problem in an inhomogeneous medium under the slow diffusion regime, as well as the asymptotic behavior of self-similar solutions.

References

- [1] M.M. Aripov, *Standard Equation's Methods for Solutions to Nonlinear problems*, (Mono-graph), FAN, Tashkent, Russian 1988.
- [2] Z.Q. Wu, J.N. Zhao, J.X. Yin, H.L. Li, *Nonlinear Diffusion Equations*, Singapore, World Sci-entific, 2001.
- [3] K. Deng, H. A. Levine, *The role of critical exponents in blow-up theorems*, The sequel, J. Math. Anal. Appl. 243 (2000), 85-126.
- [4] V.A. Galaktionov, H.A. Levine, *On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary.*, Israel J. Math. 94, 125–146 (1996).
- [5] Z.R. Rakhmonov, *On the Properties of Solutions of Multidimensional Nonlinear Filtration Problem with Variable Density and Nonlocal Boundary Condition in the Case of Fast Diffusion.*, Journal of Siberian Federal University. Mathematics Physics 2016, 9(2), 225-234.
- [6] Z. Rakhmonov, A. Zaripova, *Asymptotic Properties and Numerical Results of Solutions for a System of Degenerate Parabolic Equations with Nonlinear Boundary Conditions*, Uzbek Mathematical Journal, Vol. 70 No. 2 (2026).

Solution semi-linear Helmholtz Equation with a Quadratic Source Term for Groundwater Flow

Abdujabar Rasulov, Gulnora Raimova

*University of World Economy and Diplomacy, 54, Mustaqillik avenue,
Tashkent, Uzbekistan*

(e-mail: asrasulov@gmail.com, raimova27@gmail.com)

The boundary value problems for semi-linear Helmholtz equation use in hydrogeology to describe steady-state groundwater flow in leaky aquifers, where interaction between groundwater and soil water (or surface water) occurs [1,2]. In this context, it models the exchange of water between a shallow aquifer and an overlying source (like surface water or infiltrating soil water). This work devoted to construct a probabilistic model for solving boundary value problems for the semi-linear (Quadratic non-linear) Helmholtz equation. Probabilistic representations of problem solving in the form of the mathematical expectation of some random variable are obtained. In accordance with probabilistic representation, branching processes are constructed and modeling formulas are specified for each of the branching processes. It has been proven that the constructed branching processes degenerate with probability one and the average number of particles of each n -th generation is less than or equal to one [3,4]. For the considered problems, unbiased estimates were constructed on the trajectories of the corresponding random processes. Unbiased estimates which constructed on the trajectories of a branching process with a limited average number of branches, easily modeled in computer and has limited variance [5,6]. Based on the proposed estimates, computational experiments were carried out. The results of the computational experiment show that using the constructed algorithm it is possible to solve semi-linear problems that occur in practice.

References

- [1] Bear J., A.H.D. Cheng. Modeling groundwater flow and contaminant transport. Springer 2010, DOI: 10.1007/978-1-4020-6682-5 M. Kac, Probability and Related Topics in Physical Science, American Mathematical Soc., 1959.
- [2] Ghanmi A., Alzumi H., Zeddini N., A sub-super solution method to continuous weak solutions for a semi-linear elliptic boundary value problem on

- bounded and unbounded domains. Electronic Research Archive, 2024, 32(6): p.3742-3757., doi: 10.3934/era.2024170
- [3] Rasulov A.S., Monte Carlo algorithms for the solution Quasi-Linear Dirichlet boundary value problems of elliptical type, Mathematics and Statistics, 11(3),p.592-597, (2023). <https://www.doi.org/10.13189/ms.2023.110317>
- [4] Rasulov A, Raimova G, Bakoev M., Solution of some semi-linear Dirichlet problem by Monte Carlo method, 2020, AIP Conference Proceedings, <https://doi.org/10.1063/5.0026724>
- [5] Lupascu Q., Stanclulescu V., Numerical Solution for the Non-linear Dirichlet Problem of a Branching Process, Complex Analysis and Operator Theory, 2017, v. 11, p. 1895–1904, DOI 10.1007/s11785-017-0642-z
- [6] Beznea L., Teodor A., Positive solutions to semilinear Dirichlet problems with general boundary data, 2023, arXiv:2312.00126 , <https://doi.org/10.48550/arXiv.2312.00126>

Modeling the Instability of the Oil-Water Displacement Front During Multiphase Flow Based on a Polynomial Dynamical System

Azizaga Shakhverdiev, Yury Shestopalov

*Sergo Ordzhonikidze Russian State Geological Prospecting University,
Miklouho-Maclay St. 23., Moscow, 117997, Russian Federation
(e-mail: ah_shah@mail.ru)*

A state of unstable equilibrium in a broad class of dynamical systems (DSs) is extremely sensitive to initial conditions. In the late 19th century, Henri Poincaré demonstrated that the instability of such systems gives rise to fundamentally new problems - problems whose solutions lie beyond the scope of the Newtonian dynamics established three centuries earlier. It turned out that, in this context, the temporal evolution of an individual trajectory originating from a specific point in phase space can be replaced by a statistical or probabilistic description - specifically, by a cloud or ensemble of particles characterized by varying initial conditions (initial sets of the dynamical system, as illustrated by Fig. 1)), which furthermore undergo changes in shape as they evolve over time.

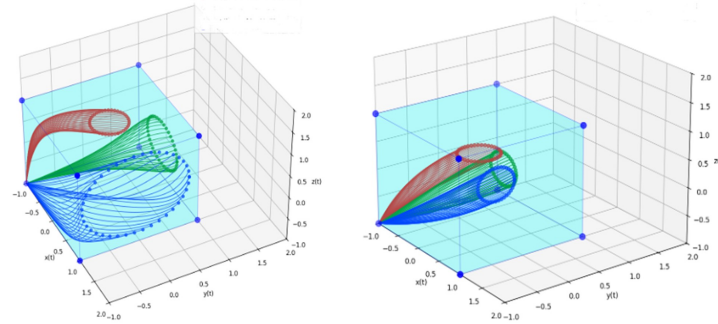


Fig. 1. Evolution of initial sets for a splitting quadratic DS, defined as circles within a characteristic parallelepiped whose vertices coincide with the DS equilibrium points. The attractor is located at the leftmost vertex.

A new physics of populations or ensembles has emerged from the work of I. R. Prigogine; this approach replaces stable dynamical systems and reversible processes with more realistic, unstable DS, the states of which are governed by randomness, fluctuations, bifurcations and dissipative structures [1]. The deterministic approach to solving complex problems essentially drives the unrestricted use of idealizations, constraints, and mathematical approximations - often disregarding the physical essence of the process - thereby simplifying the solution to the level of stable, reversible, and linear problems. As is well known, the Buckley-Leverett theory of isothermal two-phase incompressible fluid flow raises numerous questions and doubts among specialists and scientists due to the incorrectness of the results obtained - specifically, the emergence of a discontinuity and the non-uniqueness of the water saturation parameter [2]. The proposed new solution method - utilizing the theory of separable (splitting) autonomous quadratic polynomial DSs for processes occurring under conditions of unstable equilibrium - has made it possible to solve the practical problem at hand and to define the role of the control parameter [3-6]. The discriminants $D_{O,W,G}$ of the quadratic polynomial DS were defined as control parameters for each of the phases - oil, gas, and water. Using the absolute values of these discriminants in conjunction with their signs made it possible to diagnose and predict premature, catastrophic water and gas breakthrough into oil-producing wells, as exemplified in Fig. 2. Presented below are four fragments from the 27 classes of the discriminant criterion for three-phase flow, along with the corresponding well operating mode optimization strategies, formulated as follows:

1. At $D_O < 0$, $D_W > 0$, $D_G > 0$ (solution family D^{-++}), oil production shows an upward trend, while water and gas production show a downward trend.
2. At $D_O > 0$, $D_W < 0$, $D_G < 0$ (solution family D^{+--}), oil production shows a downward trend, while water and gas production are rising; displacement front instability can trigger water and gas breakthrough.
3. At $D_O < 0$, $D_W < 0$, $D_G < 0$ (solution family D^{---}), oil, water and gas productions show an upward trend.
4. At $D_O > 0$, $D_W > 0$, $D_G > 0$, (solution family D^{+++}), oil, water and gas productions show an downward trend.

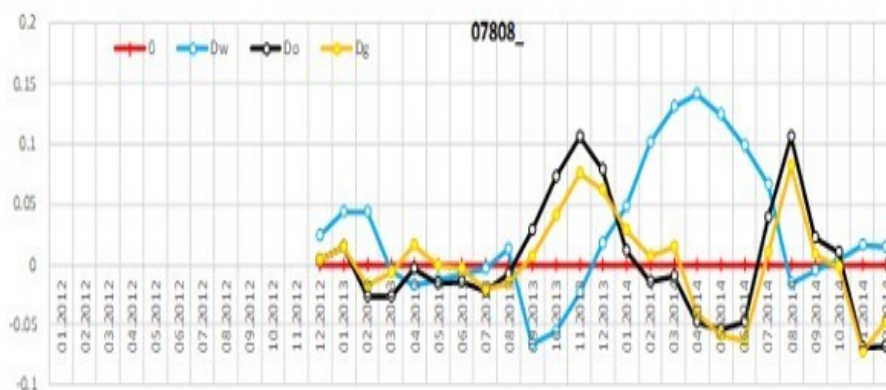


Fig. 2. Dynamics of oil, gas, and water production and dynamics of the three-phase flow discriminant for well No. 07808.

References

- [1] I. Prigogine, *The End of Certainty: Time, Chaos, and the New Laws of Nature*, Scientific Publishing Center “Regular and Chaotic Dynamics”, Izhevsk, 2000.
- [2] S.E. Buckley, M.C. Leverett, *Mechanism of fluid displacement in sands*, Society of Petroleum Engineers, 1942. <https://doi.org/10.2118/942107-G>.

- [3] Y.V. Shestopalov, A.Kh. Shakhverdiev, *Three-phase dynamical systems and their applications to monitoring for the development of hydrocarbon fields*, Lobachevskii Journal of Mathematics, 2023, 44(11), 5045–5056.
- [4] Y. Shestopalov, A. Shakhverdiev, S.V. Arefiev, *Bifurcations associated with three-phase polynomial dynamical systems and complete description of symmetry relations using discriminant criterion*, Symmetry, 2024, 16(1), 14.
- [5] A.Kh. Shakhverdiev, A.V. Denisov, V.D. Tumanova, *New optimization criteria of waterflood patterns preventing premature water breakthrough in the context of water–oil displacement front instability*, 7th Scientific Exploration Conference—Tyumen 2021: Natural Resources Management as a Cross-Functional Process, 2021.
- [6] A.Kh. Shakhverdiev, S.V. Arefiev, A.V. Denisov, R.R. Yunusov, *Method for restoring the optimal mode of operation of the reservoir–well system, taking into account the instability of the displacement front*, Oil Industry, 2020, (6), 52–57. <https://doi.org/10.24887/0028-2448-2020-6-52-57>.

Numerical Solution of Two-Parameter Singularly Perturbed Boundary Value Problems Using the Spectral-Grid Method

Sardorbek Murodov, Elvira Shakayeva, Chori Normurodov

Termez State University, Termez, Uzbekistan
(e-mail: smurodov870@gmail.com, shakayevae@gmail.com,
ch.normurodov@gmail.com)

The solutions of boundary value problems for two-parameter singularly perturbed differential equations typically exhibit the presence of two boundary layers. Due to these layers, conventional numerical methods often fail to provide accurate and efficient solutions for such problems. In the present work, a spectral-grid method is proposed for the numerical solution of this class of problems. Problems of this type are generally referred to as two-parameter singularly perturbed boundary value problems [1]. Numerous studies have

been devoted to the numerical solutions of these particular type. For example, a B-spline collocation method [2] on a similar mesh was developed and demonstrated its second-order convergence. Although several methods have been developed to solve such types of problems, there is still a lack of numerical approaches that ensure high accuracy for singularly perturbed problems involving small parameters. Therefore, in this paper, a spectral-grid method is proposed for the numerical solution of two-parameter singularly perturbed differential equations [3, 4].

Statement of the problem

Consider the following two-parameter singularly perturbed differential equation with small parameters in the highest derivatives:

$$-\varepsilon u''(x) - \mu u'(x) + u(x) = e^{1-x}, \quad x \in [0, 1], \quad (1)$$

subject to the boundary conditions

$$u(0) = u(1) = 0, \quad (2)$$

where ε and μ are small positive parameters.

To numerically model the differential problem (1)-(2), we employ the spectral-grid method based on the first-kind Chebyshev polynomials as basis functions. Typically, when analyzing the convergence and accuracy of numerical methods, an exact analytical or test function is used for verification [4]. According to this approach, an auxiliary analytical function $u_e(x)$ satisfying the boundary conditions (2) is chosen. Substituting it into equation (1) allows us to compute the corresponding right-hand side function $f(x)$. The considered problem is then solved using the spectral-grid method, and the obtained numerical solution is compared with the chosen analytical test function.

Discussion of Results

The singularly perturbed problem (1)-(2) was solved using the spectral-grid method [3]. Table 1 presents the computed values of the maximum absolute error at the Chebyshev collocation nodes. The maximum absolute error is evaluated using the following expression:

$$\Delta = \max_{0 \leq l \leq p_j} \left| u_e(x_l^{(j)}) - u_a(x_l^{(j)}) \right|, \quad j = 1, 2, \dots, N, \quad (3)$$

where u_e and u_a denote the exact and approximate solutions, respectively.

The nodes of the first-kind Chebyshev polynomials are defined by

$$x_l^{(j)} = \cos\left(\frac{\pi l}{p_j}\right), \quad l = 0, 1, \dots, p_j, \quad j = 1, 2, \dots, N. \quad (4)$$

Furthermore, let $0 < \varepsilon, \mu \ll 1$ be the perturbation parameters. The total number of Chebyshev polynomials used in the approximation is given by $M = \sum_{j=1}^N (p_j + 1)$, where p_j ($j = 1, 2, \dots, N$) denotes the number of Chebyshev polynomials employed in the j th subinterval.

From the results presented in Table 1, it can be observed that increasing the number of grid elements, as well as refining the grid near the boundaries or boundary layers, leads to a noticeable accumulation of Chebyshev nodes. This refinement results in a significant reduction of the maximum absolute error.

Table 4: Maximum absolute error of the spectral-grid method on a uniform partition.

$2^*\varepsilon$	$2^*\mu$	$2^*M = \sum_{j=1}^N (p_j + 1)$	Spectral-grid method				
			$p_1 + 1$	$p_2 + 1$	$p_3 + 1$	$p_4 + 1$	Δ
$9 \cdot 10^{-6}$	$9 \cdot 10^{-8}$	3*64	16	16	16	16	1.0×10^0
			24	8	8	24	1.8×10^{-1}
			28	4	4	28	6.9×10^{-2}
		3*128	32	32	32	32	2.2×10^{-2}
			48	16	16	48	9.2×10^{-5}
			56	8	8	56	2.9×10^{-6}
		3*256	64	64	64	64	6.1×10^{-8}
			96	32	32	96	2.4×10^{-14}
			112	16	16	112	4.5×10^{-14}

References

- [1] A. Kaushik, A. Gupta, *Adaptive mesh generation and higher-order difference approximation for the system of singularly perturbed reaction–diffusion problem*, Partial Differential Equations in Applied Mathematics, 2024. <https://doi.org/10.1016/j.padiff.2024.100750>.
- [2] M.K. Kadalbajoo, A.S. Yadaw, *B-Spline collocation method for a two-parameter singularly perturbed convection–diffusion boundary value problems*, Applied Mathematics and Computation, 201(1–2), 2008, 504–513. <https://doi.org/10.1016/j.amc.2007.12.038>.

- [3] Ch.B. Normurodov, B.A. Tursunova, *Numerical modeling of the boundary value problem of an ordinary differential equation with a small parameter at the highest derivative by Chebyshev polynomials of the second kind*, Results in Applied Mathematics, 19, 2023, 1–6. <https://doi.org/10.1016/j.rinam.2023.100388>.
- [4] Ch.B. Normurodov, Sh.A. Ziyakulova, S.K. Murodov, *On one highly accurate and efficient method for solving the biharmonic equation*, International Journal of Applied Mathematics, 38(4), 2025, 437–453. <https://doi.org/10.12732/ijam.v38i4.1>.

Explicit Inversion Formula and Regularization for Function Reconstruction from Circular Arc Integrals

Nietbay Uteuliev^a, Gafur Djaykov^b, Gulnaz Mamutova^c

^{a,b}*Nukus State Technical University, Nukus, Uzbekistan*

^c*Nukus State Pedagogical Institute, Nukus, Uzbekistan*

(e-mail: gafur_djaykov@mail.ru)

Inverse and ill-posed problems constitute an important branch of modern mathematical physics and applied mathematics. Their theory provides the mathematical foundation for numerous imaging techniques, including computerized tomography, seismic imaging, hydroacoustics, and non-destructive testing, where an unknown function must be reconstructed from indirect and often noisy measurements. The theory of regularization for ill-posed problems was established by Tikhonov [1], who proposed stable approximation methods for unstable inverse problems, while Lavrentiev [2] developed the concept of conditional well-posedness under a priori constraints.

Let

$$L_H = \mathbb{R} \times [0, H], \quad H > 0,$$

and let $u = u(x, y)$ be an unknown function. For $x \in \mathbb{R}$ and $0 < 2r \leq H$, define the circular arc

$$\Upsilon(x, r) = \{(\xi, \eta) \in L_H : (x - \xi)^2 + (r - \eta)^2 = r^2, 0 \leq \eta \leq 2r \leq H\}.$$

We consider the circular arc transform

$$\int_{\Upsilon(x,r)} u(\xi, \eta) d\sigma = f(x, r), \quad (1)$$

where f is the measured data and $d\sigma$ denotes the arc-length element.

Applying the Fourier transform with respect to the variable x , introducing the new variable $y = 2r$, and performing an Abel-type transformation, equation (1) is reduced to a Volterra integral equation of the second kind. Since Volterra equations possess a triangular structure, this reduction immediately implies uniqueness of the solution in the class of sufficiently smooth functions.

Define

$$\hat{\varphi}(\lambda, y) = \frac{1}{y} \hat{f}\left(\lambda, \frac{y}{2}\right),$$

and

$$\hat{q}(\lambda, \tau) = \frac{1}{2} \hat{\varphi}(\lambda, \tau) + \tau \frac{\partial \hat{\varphi}(\lambda, \tau)}{\partial \tau}.$$

Theorem 1. Assume that the measured data f are sufficiently smooth and satisfy the compatibility conditions ensuring that the function $\hat{q}$ is well defined and belongs to L_2 . Then the circular arc transform (1) admits at most one solution. Moreover, the Fourier transform of the solution is represented by

$$\hat{u}(\lambda, y) = \frac{2}{\pi\sqrt{y}} \int_0^y \frac{\cosh\left(\lambda\sqrt{y(y-\tau)}\right)}{\sqrt{y-\tau}} \hat{q}(\lambda, \tau) d\tau. \quad (2)$$

Formula (2) reveals the ill-posed nature of the inversion problem. Indeed, the factor

$$\cosh\left(\lambda\sqrt{y(y-\tau)}\right)$$

grows exponentially as $|\lambda| \rightarrow \infty$, leading to a severe amplification of high-frequency perturbations in the measured data. Consequently, direct inversion is unstable and requires regularization.

For $\alpha > 0$, we introduce the Gaussian regularized reconstruction

$$\hat{u}_\alpha^\delta(\lambda, y) = \frac{2e^{-\alpha^2\lambda^2}}{\pi\sqrt{y}} \int_0^y \frac{\cosh\left(\lambda\sqrt{y(y-\tau)}\right)}{\sqrt{y-\tau}} \hat{q}^\delta(\lambda, \tau) d\tau. \quad (3)$$

Let

$$\Omega_{H,y_0} = \{(x, y) : x \in \mathbb{R}, y_0 \leq y \leq H\}, \quad 0 < y_0 < H.$$

Theorem 2. Suppose that

$$u_{xx} \in L_2(\Omega_{H,y_0}),$$

and the noisy data satisfy

$$\|q^\delta - q\|_{L_2} \leq \delta_q.$$

Then, for every $\alpha > 0$, the regularized solution (3) satisfies the estimate

$$\|u_\alpha^\delta - u\|_{L_2(\Omega_{H,y_0})} \leq \frac{4\sqrt{H}}{\pi\sqrt{y_0}} \exp\left(\frac{H^2}{4\alpha^2}\right) \delta_q + \alpha^2 \|u_{xx}\|_{L_2(\Omega_{H,y_0})}. \quad (4)$$

References

- [1] V. Y. Arsenin, *On ill-posed problems*, Russian Mathematical Surveys, 1976, 31(6), 93-107.
- [2] M. M. Lavrentev, V. G. Romanov, S. P. Shishatski, *Ill-posed problems of mathematical physics and analysis*, American Mathematical Soc., 1986.

Reconstruction Algorithms for a New Class of Weighted Integral Geometry Problems with Weight Function

Nietbay Uteuliev^a, Abat Seidullaev^b, Ilkhom Bektemirov^c

^a*Nukus State Technical University, Nukus, Uzbekistan*

^b*Karakalpak State University, Nukus, Uzbekistan*

^c*Nukus State Pedagogical Institute, Nukus, Uzbekistan*

(e-mail: abat1984@gmail.com)

The mathematical model studied in this work is closely connected with inverse problems arising in seismic imaging, where wave propagation in layered media is approximated by piecewise linear ray trajectories due to reflection and refraction. Related models also appear in X-ray and emission tomography [1],

photoacoustic and thermoacoustic imaging[2], Compton scattering tomography[3], ultrasound and optical tomography[4], as well as non-destructive testing. In all these applications, the measured data are represented by weighted integral transforms of an unknown function defined over prescribed families of curves.

Let $L_H = \mathbb{R} \times (0, H)$, with $H > 0$, and assume that the unknown function $u = u(x, y)$ is an element of the Sobolev space $W_2^2(L_H)$. For each point $(x, y) \in L_H$, we define the broken-line trajectory:

$$S(x, y) = \{(\xi, \eta) \in L_H : |x - \xi| = y - \eta, \quad 0 < \eta \leq y\},$$

which consists of two straight-line segments issuing from the point (x, y) .

Let the weighted broken-ray transform of the unknown function u be defined by

$$\int_{S(x,y)} (y - \eta) u(\xi, \eta) ds = f(x, y), \quad (1)$$

where $f(x, y)$ represents the measured transform and ds denotes the differential arc length along the curve $S(x, y)$. The corresponding inverse problem is to determine the function $u(x, y)$ in L_H from the knowledge of the transform $f(x, y)$.

Theorem 1. *Assume that $f \in W_2^4(L_H)$. Then equation (1) has a unique solution $u \in W_2^2(L_H)$. This solution is characterized by the Poisson equation*

$$\Delta u(x, y) = \frac{\sqrt{2}}{4} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right)^2 f(x, y),$$

where Δ is the Laplace operator. In addition, the solution depends continuously on the data in the sense that

$$\|u(x, y)\|_{W_2^2} \leq C \|f(x, y)\|_{W_2^4}$$

where $C > 0$ is a positive constant independent of the function f .

After deriving the explicit inversion formula, the inverse problem is reduced to the solution of a Poisson-type equation. Although this equation admits an analytical representation, different reconstruction techniques exhibit different numerical characteristics in terms of stability, computational efficiency, and sensitivity to measurement noise. Therefore, it is reasonable to investigate several reconstruction algorithms based on the same mathematical model.

References

- [1] F. Natterer, F. Wübbeling, *Mathematical methods in image reconstruction* – Society for Industrial and Applied Mathematics, 2001.
- [2] P. Kuchment, L. Kunyansky, *Mathematics of photoacoustic and thermoacoustic tomography* arXiv preprint arXiv:0912.2022. – 2009.
- [3] G. Ambartsoumian, *V-line and conical Radon transforms with applications in imaging*, Chapter 7 in “The Radon Transform: The First 100 Years and Beyond”, edited by R. Ramlau and O. Scherzer, Radon Series on Computational and Applied Mathematics. – 2019.
- [4] S. J. Norton, M. Linzer, *Ultrasonic reflectivity imaging in three dimensions: exact inverse scattering solutions for plane, cylindrical, and spherical apertures* IEEE Transactions on biomedical engineering. 1981, 2. 202-220.

Scheme Based Comparison for Benjamin-Bona-Mahony-Burgers Equation Governing Flood Predictions

Priyanka Yadav, Sumita Dahiya

*Department of Mathematics, Netaji Subhas University of Technology,
Delhi-110078, India*

(E-mail: priyanka.phd23@nsut.ac.in)

This manuscript aims to assess the applicability of a variant of the cubic B-spline collocation method to handle nonlinear partial differential equations of higher orders, usually more than two. For this purpose, two different versions of the Benjamin–Bona–Mahony–Burgers equation are taken into account and numerically simulated using the proposed method. The time derivatives in the governing equation are approximated by employing the Crank–Nicolson scheme, while the improvised cubic B-spline collocation method is designed to approximate the spatial derivatives. The nonlinearity in the governing equation has been dealt with using the Rubin–Graves method. The techniques that

are employed ensure that the resulting matrix remains diagonally dominant and the boundary conditions are implemented with ease. The von Neumann analysis technique is used to establish stability. The conserved quantities are calculated to verify that the conservation laws are maintained as expected from the theory. The numerical results confirm the accuracy and effectiveness of the approach for both versions of the BBM–Burgers equation and confirm its suitability for solving nonlinear problems of higher order in both space and time.

Keywords: Flood prediction, Benjamin–Bona–Mahony–Burgers equation, improvised cubic B-spline collocation approach, Crank–Nicolson scheme, Rubin–Graves method, von Neumann analysis.

References

- [1] Bona, J. L., W. G. Pritchard, and L. R. Scott. "An evaluation of a model equation for water waves." *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 302, no. 1471 (1981): 457-510.
- [2] Benjamin, Thomas Brooke, Jerry Lloyd Bona, and John J. Mahony. "Model equations for long waves in nonlinear dispersive systems." *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 272, no. 1220 (1972): 47-78.
- [3] Dahiya, Sumita, Aditi Singh, and S. P. Singh. "Study of the gardner equation with homogeneous boundary conditions via fourth order modified cubic b-spline collocation method." *Computational Mathematics and Mathematical Physics* 63, no. 12 (2023): 2474-2491.
- [4] Dehghan, Mehdi, Nasim Shafieeabyaneh, and Mostafa Abbaszadeh. "Numerical and theoretical discussions for solving nonlinear generalized Benjamin–Bona–Mahony–Burgers equation based on the Legendre spectral element method." *Numerical Methods for Partial Differential Equations* 37, no. 1 (2021): 360-382.

Refinement of Low-Resolution Images with High-Resolution Images

Zakir Zabidov, Khayala Gadirova

Institute of Mathematics, Baku, Azerbaijan

(e-mail: zakir_zabidov@mail.ru; memmedagaqizi@gmail.com)

When using remote sensing methods, as a rule, the characteristics associated with the object are measured, not the characteristics of the object being studied. [1,2,3]. It should be noted that there are a large number of pre-processing methods and algorithms for remotely sensed data. [4,5,6] The rapid development of computer science and the wide capabilities of various software systems (e.g. Matlab, Mathematica, Maple, etc.) have made it possible to use satellite data in various identification and recognition problems or have stimulated the improvement of the accuracy of solution algorithms. The accuracy of the solution to the problem usually depends on many factors. Some of these factors are mentioned in the present work. Also, this paper presents a method for refining low-resolution images with high-resolution images. In the present work, for simplicity, it is assumed that one pixel in a low-resolution image covers 4 (four) pixels in a high-resolution image. Here, low and high resolution images of the same area selected for testing are shown in RGB format respectively. Here, the rule for mapping any pixel (i, j) of a low-resolution image to pixels of a high-resolution image is given. The determination of the brightness values in the pixel determined by the inversion rule was performed by applying a neural network. In the neural network under consideration, the spectral values of the brightness value of each of the four pixels associated with it (adjacent) and taken as input data were taken. The "logsig" activation function was also used in this neural network. The brightness values in the image obtained using the neural network were compared with the brightness values of a previously taken low-resolution image. It should be noted that the comparison between brightness values was performed in the BLUE, GREEN and RED spectral channels. It is shown that the number of pixels containing errors (difference between brightness values greater than 15) does not exceed 3% of the total number of pixels

Acknowledgment. This research was supported by the Azerbaijan Science Foundation (Grant No. AEF-MGC-2025-1(54)-20/02/1-M-02).

References

- [1] O.S. Tokareva, *Processing and Interpretation of Remote Data Sensing*, Publishing House of Tomsk Polytechnic University, Tomsk, 2010, 148 p.
- [2] V.T. Fisenko, T.Yu. Fisenko, *Computer Processing and Image Recognition*, St. Petersburg State University ITMO, St. Petersburg, 2008, 192 p.
- [3] *Space Monitoring in Agriculture*, SOVZOND Company, Moscow, 34 p.
- [4] E.I. Karavanova, *Optical Properties of Soil and Their Nature*, Publishing House of Moscow State University, Moscow, 2003, 153 p.
- [5] V.B. Kashkin, *Digital Processing of Aerospace Images*, IPK SFU, Krasnoyarsk, 2008, 278 p.
- [6] *MODIS Technology in Terra and Aqua Satellites*, NASA .
- [7] V.P. Dyakonov, *MATLAB: Complete Tutorial*, DMK Press, Moscow, 2012, 768 p.
- [8] A.B. Sergienko, *Digital Signal Processing*, Piter, St. Petersburg, 2002, 608 p.
- [9] R. Gonzales, R. Woods, *Digital Image Processing*, Techno-Sphere, Moscow, 2012, 1104 p.
- [10] V.S. Medvedev, V.G. Potemkin, *Neural Networks. MATLAB 6*, 2001, 630 p.
- [11] V. Dyakonov, V. Kruglov, *MATLAB Extension Packages. Special Reference Manual*, 2001, 480 p.

A-Tomographic Transform and Main Properties

Nasridin Zhabborov^a, Behzod Husenov^b, Abdurasul Narmanov^c

^a*Islam Karimov str. 49, 100066, Tashkent, Uzbekistan*

^b*Muhammad Ikbal str. 11, 200117, Bukhara, Uzbekistan*

^c*Yogdu str. 1, 100175, Tashkent, Uzbekistan*

(e-mail: zhabborovnasridin@gmail.com, b.e.husenov@buxdu.uz,
a-narmanov@mail.ru)

The Radon transform, which assigns to a function its integrals over straight lines, plays a fundamental role in integral geometry, inverse problems, and mathematical physics. Its classical theory forms the analytical basis of X-ray tomography and has led to numerous generalizations involving integration over more general families of curves and surfaces. Such generalized transforms arise naturally in applications ranging from medical imaging and geophysics to non-destructive testing and scattering theory [1]–[3].

First, let's give definitions and basic properties:

Definition 1. Let $A : \Omega \rightarrow \mathbb{C}$ be a holomorphic function, where $\Omega \subset \mathbb{C}$ is a domain. For $\tau \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$ define the family of level curves

$$\Gamma_{(\tau, \varphi)} = \{z \in \Omega : \operatorname{Re}(e^{-i\varphi} A(z)) = \tau\}. \quad (1)$$

For a function $f \in C_0^\infty(\Omega)$, the $A(z)$ -tomographic transform is defined by

$$T_A f(\tau, \varphi) := \int_{\Gamma_{(\tau, \varphi)}} f(z) ds, \quad (2)$$

where ds denotes the Euclidean arc length element along $\Gamma_{(\tau, \varphi)}$.

Remark 1. If $A(z) = z$, then the family $\Gamma_{(\tau, \varphi)}$ reduces to the family of straight lines in the plane, and T_A coincides with the classical Radon transform. For nonlinear choices of $A(z)$, the family $\Gamma_{(\tau, \varphi)}$ gives rise to geometrically nontrivial level curves, including lemniscate-type configurations.

Regularity assumptions and injectivity

Assumption 1. We assume that the family of curves $\Gamma_{(\tau, \varphi)}$ is regular in the following sense:

- (i) for each the (τ, φ) set $\Gamma_{(\tau, \varphi)}$ is a smooth curve in Ω ;

(ii) the parametrization of the family $(\tau, \varphi) \rightarrow \Gamma_{(\tau, \varphi)}$ is nondegenerate;
 (iii) the associated canonical relation of the transform T_A satisfies the Bolker-type condition. Under these assumptions, the operator T_A can be treated as a Fourier integral operator of Radon type.

Theorem 1. Under Assumption 1, the $A(z)$ -tomographic transform

$$T_A : C_0^\infty(\Omega) \rightarrow C_0^\infty(\mathbb{R} \times [0, 2\pi))$$

is injective. More precisely, if $T_A f(\tau, \varphi) = 0$ for all $(\tau, \varphi) \in \mathbb{R} \times [0, 2\pi)$, then $f \equiv 0$ in Ω .

Normal operator and ellipticity Lemma 1. Under Assumption 1, the normal operator $N_A = T_A^* T_A$ is a pseudo-differential operator of order -1 on Ω . Moreover, is elliptic. Its principal symbol has the form

$$\sigma(N_A)(z, \xi) \sim c_A(z, \xi) |\xi|^{-1} \quad (3)$$

where $c_A(z, \xi)$ is a smooth non-vanishing function on the relevant cotangent bundle.

Corollary 1. Under Assumption 1, there exists a pseudo-differential operator $Q \in \Psi^1(\Omega)$ such that

$$QN_A = I + K,$$

where K is a smoothing operator. This follows from the standard elliptic parametric construction for N_A .

References

- [1] Helgason S., *The Radon Transform*, 2nd ed. Birkhauser, Boston, 1999.
- [2] Natterer F., *The Mathematics of Computerized Tomography*, SIAM, Philadelphia, 2001.
- [3] Quinto E.T., *Singularities of the X-ray transform and limited data tomography in $\mathbb{R}^2$ and $\mathbb{R}^3$* , SIAM Journal on Mathematical Analysis, 24(5) (1993), 1215-1225.
- [4] A.Sadullayev, N.M.Zhabborov, *On a class of A -analytic functions*, Journal Siberian Federal University. Mathematics and Physics, 2016, 9(3), 374-383.

DIFFERENTIAL EQUATIONS AND OPTIMAL CONTROL**Feedback Control of Boundary Conditions and Sensor Motion During Rod Heating****Vagif Abdullayev^{a,b}, Vugar Hashimov^a**^a*Institute of Mathematics, Baku, Azerbaijan*^b*Azerbaijan State Oil and Industry University, Baku, Azerbaijan*

(e-mail: vaqif_ab@rambler.ru)

The paper sets and studies the problem of synthesizing the control of the power of heat sources located at the ends of a rod using the values of current and past measurements. It is necessary to determine the optimal constant values of the coefficients involved in the formulas for these dependencies. These coefficients act as feedback parameters. Determining the values of these coefficients leads to the minimization of a finite-dimensional function in the space of constant values of the feedback parameters. Gradient type methods were used to minimize the function. For this purpose, analytical formulas for the gradient components of the objective function were obtained. The proposed approach is illustrated with examples of solving test problems.

Let us consider the IBVP of controlling the heating process of a rod:

$$u_t(x, t) = a^2 u_{xx}(x, t) - \lambda_0 [u(x, t) - \theta], \quad x \in (0, l), \quad t \in (0, T], \quad (1)$$

$$u(x, 0) = b(x) = b = \text{const}, \quad x \in [0, l], \quad (2)$$

$$u_x(0, t) = \lambda_1 (u(0, t) - \vartheta_1(t)), \quad t \in (0, T], \quad (3)$$

$$u_x(l, t) = -\lambda_2 (u(l, t) - \vartheta_2(t)), \quad t \in (0, T]. \quad (4)$$

Here $u(x, t)$ denotes the rod temperature at point x at time t ; θ is the ambient temperature; T is the process duration; the coefficients a^2 , λ_0 , λ_1 , λ_2 are given.

The control $\vartheta_1(t)$ and $\vartheta_2(t)$ are piecewise continuous functions and satisfy to technological constraints:

$$\underline{\vartheta}_1 \leq \vartheta_1(t) \leq \overline{\vartheta}_1, \quad \underline{\vartheta}_2 \leq \vartheta_2(t) \leq \overline{\vartheta}_2, \quad t \in [0, T].$$

and the parameters $\underline{\vartheta}_1$, $\overline{\vartheta}_1$, $\underline{\vartheta}_2$, $\overline{\vartheta}_2$ are given.

The rod temperature is uniform along its length but not exactly known $b \in \mathcal{B}$ describes its possible values with density $\rho_{\mathcal{B}}(b) \geq 0$. The ambient temperature is constant, $\theta(t) = \theta = \text{const}$, $t \in [0, T]$ is also unknown and described by a set $\theta \in \Theta$ with density $\rho_{\Theta}(\theta) \geq 0$, where

$$\int_{\mathcal{B}} \rho_{\mathcal{B}}(b) db = 1 \quad \int_{\Theta} \rho_{\Theta}(\theta) d\theta = 1.$$

It is required to determine the control action $\vartheta = \vartheta(t) = (\vartheta_1(t), \vartheta_2(t))$, at which the given functional reaches its minimum, at the final moment of time from the given desired temperature distribution along the rod $U(x)$, $x \in [0, l]$ over all permissible external temperatures θ and initial conditions b :

$$J(\vartheta) = \int_{\Theta} \int_{\mathcal{B}} I(\vartheta; b, \theta) \rho_{\mathcal{B}}(b) \rho_{\Theta}(\theta) db d\theta, \quad (5)$$

$$I(\vartheta; b, \theta) = \int_0^l \mu(x) [u(x, T; \vartheta, b, \theta) - U(x)]^2 dx + \varepsilon \|\vartheta(t) - \hat{\vartheta}\|_{L_2[0, T]}^2. \quad (6)$$

Here $u(x, t; \vartheta, b, \theta)$ is the solution of the IBVP (1)–(4) for the admissible initial condition b and ambient temperature θ ; ε , $\hat{\vartheta}$ are the regularization parameters of the functional; $\mu(x) \geq 0$, $x \in [0, l]$ is the given weight function.

Let there be N moving sensors measuring the temperature along the rod to provide boundary control feedback. The movements is controlled by Cauchy problems defined by nonlinear differential equations [1, 2]:

$$\begin{aligned} \dot{\xi}_i(t) &= f_i(\xi_i(t), \omega_i(t), t), \quad t \in (0, T], \quad i = 1, 2, \dots, N, \\ \xi_i(0) &= \xi_i^0 \in [0, l], \quad i = 1, 2, \dots, N. \end{aligned}$$

Here, $\xi_i(t)$ denotes the position of the i -th temperature sensor at time $t \in [0, T]$; the given functions $f_i(\cdot, \cdot, t)$, $i = 1, 2, \dots, N$ are continuously differentiable in the first two arguments and piecewise continuous in the third; the controls $\omega_i(t)$, $i = 1, 2, \dots, N$ define the optimized sensor motions; ξ_i^0 , $i = 1, 2, \dots, N$ are the initial sensor positions on the rod.

Control actions on sensor movements must satisfy technological constraints:

$$\underline{\omega}_i \leq \omega_i(t) \leq \overline{\omega}_i, \quad t \in [0, T], \quad i = 1, 2, \dots, N.$$

Let us denote by continuously measured in time by the i -th sensor temperature at point $\xi_i(t)$ at time t , $i = 1, 2, \dots, N$,

$$\hat{u}_i(t) = u(\xi_i(t), t), \quad t \in [0, T], \quad i = 1, 2, \dots, N. \quad (7)$$

The values of the boundary controls $\vartheta_1(t)$, $\vartheta_2(t)$ and the sensor motion controls $\omega_i(t)$ are determined from measurements at the current and past time moments, and are assumed to depend linearly on the process state at the controlled points:

$$\vartheta_1(t; \mathbf{k}) = \sum_{i=1}^N k_i^1 [\hat{u}_i(t) - U(t)] + k_i^4 [\hat{u}_i(t - \Delta) - U(t - \Delta)], \quad t \in [0, T],$$

$$\vartheta_2(t; \mathbf{k}) = \sum_{i=1}^N k_i^2 [\hat{u}_i(t) - U(t)] + k_i^5 [\hat{u}_i(t - \Delta) - U(t - \Delta)], \quad t \in [0, T],$$

$$\omega_i(t; \mathbf{k}) = k_i^3 [\hat{u}_i(t) - U(t)] + k_i^6 [\hat{u}_i(t - \Delta) - U(t - \Delta)], \quad t \in [0, T],$$

here, $\mathbf{k} = (k_i^1, k_i^2, k_i^3, k_i^4, k_i^5, k_i^6)$, $i = 1, 2, \dots, N$ are the gain coefficients.

In the case of continuous measurement (7) of the heating process, the minimized functional (5), (6) is written as:

$$J(\mathbf{k}) = \int_{\Theta} \int_{\mathcal{B}} I(\mathbf{k}; b, \theta) \rho_{\mathcal{B}}(b) \rho_{\Theta}(\theta) db d\theta,$$

$$I(\mathbf{k}; b, \theta) = \int_0^l \mu(x) [u(x, T; \mathbf{k}, b, \theta) - U(x)]^2 dx + \varepsilon \|\mathbf{k} - \hat{\mathbf{k}}\|_{\mathbb{R}^{6N}}^2.$$

References

- [1] V.M. Abdullayev, V.A. Hashimov, *Control of loaded points of a parabolic equation*, Proceedings of the Institute of Mathematics and Mechanics, 2024, 50(1), 21-38
- [2] V.A. Hashimov, *Synthesis of boundary conditions control and state measurement points motion for fast damping of string vibrations*, Proceedings of the Institute of Mathematics and Mechanics, 2025, 51(2), 294-317

On Stabilization of Solutions to Benjamin-Bona-Mahony-Burgers Equation

Aygun Abdullayeva, Atif Namazov

Azerbaijan Technical University, Baku, Azerbaijan
(e-mail: atif.namazov@aztu.edu.az)

We study the problem of interior stabilization of solutions to Benjamin-Bona-Mahony Burgers (BBMB) equation.

$$v_t - bv_{xxt} - \nu v_{xx} - \alpha v + vv_x + f(v) = h_0(x), x \in (0, L). \quad (1)$$

under homogenous Dirichlet's boundary conditions

$$v(0, t) = v(L, t) = 0, \quad t > 0, \quad (2)$$

where $b > 0, \nu > 0, \alpha > 0$ are given parameters $h_0 \in L^2(0, L)$ is a given source term and $f(\cdot) \in C^1(\mathbb{R})$ is a given nonlinear term that satisfies the conditions

$$f(0) = 0, f'(s) \geq 0, \quad \forall s \in \mathbb{R}$$

We establish global stabilization of solutions for the (BBMB) equation using finite parameter feedback control algorithms, which employ finitely many Fourier modes and finitely many volum elements. Specifically we discuss the problem of global stabilization to a given steady state solution of initial boundary value problem (1)-(2).

Inverse Problem for Burgers-Type Parabolic Equation

Adalat Akhundov^a, Areste Habibova^b, Nahid Pashaev^c

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Baku Engineering University, Baku, Azerbaijan*

^c*Lankaran State University, Lankaran, Azerbaijan*

(e-mail:adalatakhundov@gmail.com, ahabibova@beu.edu.az,
umud-96@mail.ru)

The paper considers the inverse problem of finding the unknown coefficient on the right-hand side of the multidimensional Burgers equation. Direct problems for the Burgers equation in one-dimensional case have been considered by [1], while those in multidimensional case have been treated by [2] and others.

We consider the following inverse problem of finding a pair of functions $\{f_k(x), u_k(x, t), x = \overline{1, n}\}$:

$$u_{kt} - \Delta u_k + \sum_{j=1}^n u_j u_{kx_j} = f_k(x) g_k(x, t), \quad (x, t) \in Q = D \times (0, T], \quad (1)$$

$$u_k(x, 0) = \varphi_k(x), x \in \overline{D} = D \cup \partial D, u_k(x, t) = \psi_k(x, t), (x, t) \in S = \partial D \times [0, T], \quad (2)$$

$$\int_0^T u_k(x, t) \theta(t) dt = h_k(x), x \in \overline{D} \quad (3)$$

where $0 < T = \text{const}$, the given functions $g_k(x, t), \varphi_k(x), \psi_k(x, t), h_k(x), \theta(t)$ satisfy the conditions:

$$\begin{aligned} \text{A. } g_k(x, t) &\in C^{\alpha, \alpha/2}(\overline{Q}), 0 < g_k^* \leq \int_0^T |g_k(x, t)| |\theta(t)| dt \leq g_k^{**}, \varphi_k(x) \in \\ &C^{2+\alpha}(\overline{D}), \psi_k(x, t) \in C^{2+\alpha, 1+\alpha/2}(S), \varphi_k(x) = \psi_k(x, 0), x \in \partial D, \\ &h_k(x) \in C^{2+\alpha}(\overline{D}), \theta(t) \in C^1([0, T]), g_k^*, g_k^{**} = \text{const}, \alpha \in (0, 1), k = \overline{1, n} \end{aligned}$$

It is proved that problem (1)-(3) is equivalent to the problem of determining $\{f_k(x), u_k(x, t), k = \overline{1, n}\}$ from conditions (1),(2) and

$$f_k(x) = \left[u_k(x, T) \theta(T) - \varphi_k(x) \theta(0) - \int_0^T (u_k(x, t) \theta_t(t) - \right.$$

$$\left. - \sum_{j=1}^n u_j(x, t) u_{kx_j}(x, t) \theta(t) \right) dt - \Delta h_k(x) \Big] / \int_0^T g_k(x, t) \theta(t) dt, \quad x \in \overline{D} \quad (4)$$

Let the pair functions $\{f_k(x) \in C^\alpha(\overline{D}), u_k(x, t) \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}), k = \overline{1, n}\}$ be the solutions to problem (1),(2),(4). Then the solution to the problem of finding $u_k(x, t)$, $k = \overline{1, n}$ from (1),(2) can be represented as follows

$$u_k(x, t) = \Phi_k(x, t) + \int_0^T \int_D G_k(x, t, \xi, \tau) [f_k(\xi) g_k(\xi, \tau) - \\ - \sum_{j=1}^n u_j(\xi, \tau) u_{kx_j}(\xi, \tau)] d\xi d\tau, \quad (x, t) \in Q, \quad (5)$$

where

$$d\xi = d\xi_1, \dots, d\xi_n, \Phi_k(x, t) = F_k(x, t) + \int_0^T \int_D G_k(x, t, \xi, \tau) [\Delta F(\xi, \tau) - F_\tau(\xi, \tau)] d\xi, d\tau \\ F_k(x, t) \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}), \quad F_k(x, 0) = \varphi_k(x), x \in \overline{D}, \quad F_k(x, t) = \psi_k(x, t), \\ (x, t) \in S, \quad G_k(x, t, \xi, \tau) \text{ the Green function of the problem}$$

$$w_{kt} - \Delta w_k = P_k(x, t), \quad (x, t) \in Q,$$

$$w_k(x, 0) = 0, x \in \overline{D}, \quad w_k(x, t) = 0, (x, t) \in S,$$

$$P_k(x, t) = f_k(x) g_k(x, t) - \sum_{j=1}^n u_j(x, t) u_{kx_j}(x, t) + \Delta F(x, t) - F_t(x, t).$$

Definition 1. We will call the functions $\{f_k(x), u_k(x, t), x = \overline{1, n}\}$ a generalized solution to problem (4),(5) if : 1) $f_k(x) \in C(\overline{D})$; 2) $u_k(x, t) \in C^{1,0}(\overline{Q})$; 3) the relations (4),(5) are satisfied for these functions.

Theorem 1. Let the data of problem (4),(5) satisfy conditions A. Then, for sufficiently small T^* , the problem (4),(5) has at least one solution in the sense of Definition 1 for $(x, t) \in \overline{Q}_* = \overline{D} \times [0, T^*]$.

The estimation of stability of the solution of inverse problems occupies a central place in the investigation of their well-posedness.

Define the following set:

$$K = \left\{ (f, u) \mid f_k(x) \in C(\overline{D}), u_k(x, t) \in C^{1,0}(\overline{Q}), \text{ for all } (f, u) \mid f_k(x), \right. \\ \left. \mid D_x^l u_k(x, t) \mid \leq c_1, l = 0, 1, k = \overline{1, n}, (x, t) \in \overline{Q}, c_1 = \text{const} \right\}.$$

Let's assume we are given two input sets $\{g_k^i(x, t), \varphi_k^i(x), \psi_k^i(x, t), h_k^i(x), \theta(t), k = 1, n, i = 1, 2\}$. We denote problem (4),(5) with data $\{g_k^i, \varphi_k^i, \psi_k^i, h_k^i, \theta(t), k = 1, n, i = 1, 2\}$ by $B_i, i = 1, 2$. Let $\{f_k^1(x), u_k^1(x, t)\}$ and $\{f_k^2(x), u_k^2(x, t)\}$ be solutions of problems B_1 and B_2 respectively.

Theorem 2. Let the following conditions holds: 1) the functions $\{g_k^i, \varphi_k^i, \psi_k^i, h_k^i, \theta, k = 1, n, i = 1, 2\}$ satisfy conditions A; 2) solutions of problems B_1 and B_2 exist in the sense of Definition 1 and belong to the set K .

Then there exist a $T_1 (0 < T_1 \leq T)$ such that for $(x, t) \in \overline{D} \times [0, T_1]$ the solution of problem (4),(5) satisfies the stability estimate

$$\|u^1 - u^2\|_Q^{(0)} + \|f^1 - f^2\|_D^{(0)} \leq c \left[\|g^1 - g^2\|_Q^{(0)} + \|\varphi^1 - \varphi^2\|_D^{(2)} + \right. \\ \left. + \|\psi^1 - \psi^2\|_S^{(2.1)} + \|h^1 - h^2\|_D^{(2)} \right],$$

Where $c > 0$ depend on data of problem B_1, B_2 and the set K ,

$$\|w\|_E^{(l,m)} = \sum_{k=1}^n \left[\sum_{j=0}^l \sup_E |D_x^j u_k(x, t)| + \sum_{i=0}^m \sup_E \left| \frac{\partial^i u_k(x, t)}{\partial t^i} \right| \right].$$

References

- [1] G.B. Whitham, *Linear and nonlinear waves world* 1977, 624 p. (In Russian)
- [2] A.E. Mamontov, Existence of global solutions of multidimensional Burgers equations of a compressible viscous fluid. Sbornik Mathematical Library functional analysis Moscow, 190(8), 1131-1150, 1999.

Investigation of the Solution of the Boundary Value Problem for a Linear, Homogeneous, Constant Coefficient Ordinary Differential Equation with Fractional Derivatives

Nihan Aliyev^a, Ramin Zeynalov^{b,c}

^a*Baku State University, Baku, Azerbaijan*

^b*Institute of Mathematics, Baku, Azerbaijan*

^c*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
(e-mail: nihan@aliev.info, raminzeynalov1304@gmail.com)

In the presented work, the solution of the problem within the nonlocal boundary condition for a third-order ordinary, linear, constant coefficient, homogeneous differential equation with step $\alpha \in (0, 1)$ is investigated

Let us consider the following problem:

$$D^{3a}y(x) + pD^{2a}y(x) + qD^ay(x) + ry(x) = 0, \quad 0 < x_1 < x < x_2, \quad (1)$$

$$\sum_{k=0}^2 \left[a_{jk}^{(1)} D^k y(x) /_{x=x_1} + a_{jk}^{(2)} D^k y(x) /_{x=x_2} \right] = a_j, \quad j = \overline{1, 3}, \quad (2)$$

where $p, q, r, a_{jk}^{(1)}, a_{jk}^{(2)}$ and a_j are given constants, $y(x)$ is the sought function.

Let us find the solution to equation (1) in the form of a parameter-dependent modified Mittag-Leffler series:

$$y(x) \equiv y(x, \rho) = \sum_{k=1}^{\infty} \rho^{k-1} \frac{x^{-1+kj}}{(-1+kj)!}, \quad (3)$$

where ρ is an unknown parameter. We differentiate (3) as follows:

$$\begin{aligned} D^a y(x) &= \sum_{k=1}^{\infty} \rho^{k-1} \frac{x^{-1+(k-1)j}}{(-1+(k-1)j)!} = \frac{x^{-1}}{(-1)!} + \rho \frac{x^{-1+\alpha}}{(-1+\alpha)!} + \rho^2 \frac{x^{-1+2\alpha}}{(-1+2\alpha)!} + \dots = \\ &= \delta(x) + \rho \left[\frac{x^{-1+\alpha}}{(-1+\alpha)!} + \rho \frac{x^{-1+2\alpha}}{(-1+2\alpha)!} + \dots \right] = \rho y(x, \rho), \end{aligned} \quad (4)$$

in the same way,

$$D^{2j}y(x) = \rho^2 y(x, \rho), \quad D^{3j}y(x) = \rho^3 y(x, \rho). \quad (5)$$

Let's write (3)-(5) equation (1):

$$\begin{aligned}\rho^3 y(x, \rho) + p\rho^2 y(x, \rho) + q\rho y(x, \rho) + ry(x, \rho) &= 0, \\ \rho^3 + p\rho^2 + q\rho + r &= 0\end{aligned}\quad (6)$$

Let us assume that there are three different real roots of equation (6). Let us denote them by ρ_1, ρ_2 and ρ_3 . ($\rho_k \in \mathbb{R}, k = \overline{1, 3}, \rho_i \neq \rho_j, i \neq j$). Then the general solution of equation (1) is of the form

$$y(x) = \sum_{k=1}^3 C_k y(x, \rho_k). \quad (7)$$

Here, C_k are arbitrary constants. We will determine the arbitrary constants included in the general solution from the boundary conditions (2):

$$\begin{aligned}\sum_{m=0}^2 \left[a_{jm}^{(1)} \sum_{n=1}^3 C_n \rho_n^m y(x_1, \rho_n) + a_{jm}^{(2)} \sum_{n=1}^3 C_n \rho_n^m y(x_2, \rho_n) \right] &= a_j, \quad j = \overline{1, 3}, \\ \sum_{m=0}^2 \sum_{n=1}^3 C_n \rho_n^m \left[a_{jm}^{(1)} y(x_1, \rho_n) + a_{jm}^{(2)} y(x_2, \rho_n) \right] &= a_j, \quad j = \overline{1, 3}.\end{aligned}\quad (8)$$

Let's accept that,

$$\Delta_{jn} = \sum_{m=0}^2 \rho_n^m \left[a_{jm}^{(1)} y(x_1, \rho_n) + a_{jm}^{(2)} y(x_2, \rho_n) \right], \quad j = \overline{1, 3}; n = \overline{1, 3}. \quad (9)$$

Then

$$\Delta = \begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{vmatrix} \neq 0, \quad (10)$$

Under condition (10), the system of linear algebraic equations (8) has a unique solution according to Cramer's rule.

$$C_1 = \frac{1}{\Delta} \begin{vmatrix} a_1 & \Delta_{12} & \Delta_{13} \\ a_2 & \Delta_{22} & \Delta_{23} \\ a_3 & \Delta_{32} & \Delta_{33} \end{vmatrix}, \quad C_2 = \frac{1}{\Delta} \begin{vmatrix} \Delta_{11} & a_1 & \Delta_{13} \\ \Delta_{21} & a_2 & \Delta_{23} \\ \Delta_{31} & a_3 & \Delta_{33} \end{vmatrix},$$

$$C_3 = \frac{1}{\Delta} \begin{vmatrix} \Delta_{11} & \Delta_{12} & a_1 \\ \Delta_{21} & \Delta_{22} & a_2 \\ \Delta_{31} & \Delta_{32} & a_3 \end{vmatrix},$$

Thus, the solution to the boundary problem (1)-(2) is as follows:

$$y(x) = \frac{y(x, \rho_1)}{\Delta} \begin{vmatrix} a_1 & \Delta_{12} & \Delta_{13} \\ a_2 & \Delta_{22} & \Delta_{23} \\ a_3 & \Delta_{32} & \Delta_{33} \end{vmatrix} + \frac{y(x, \rho_2)}{\Delta} \begin{vmatrix} \Delta_{11} & a_1 & \Delta_{13} \\ \Delta_{21} & a_2 & \Delta_{23} \\ \Delta_{31} & a_3 & \Delta_{33} \end{vmatrix} + \frac{y(x, \rho_3)}{\Delta} \begin{vmatrix} \Delta_{11} & \Delta_{12} & a_1 \\ \Delta_{21} & \Delta_{32} & a_2 \\ \Delta_{31} & \Delta_{32} & a_3 \end{vmatrix}, \quad (11)$$

Theorem. If and $p, q, r, a_{jk}^{(1)}, a_{jk}^{(2)}$ and $a_j, j = \overline{1, 3}; k = \overline{0, 2}$ are real constants given, and the boundary conditions (2) are not linearly dependent, and condition (10) is satisfied, then the boundary value problem (1)-(2) has a unique solution, and this solution is in the form (11).

Equivalence of Minimality Conditions

Narmin Aliyeva^a, Yagub Aliyev^b

^aBaku State University, Baku, Azerbaijan

^bADA University, Baku, Azerbaijan

(e-mail: mathaliyeva@gmail.com, yaliyev@ada.edu.az)

We consider the Sturm–Liouville problem

$$-y'' + q(x)y = \lambda y, \quad 0 < x < 1, \quad (1)$$

$$y(0) \cos \beta = y'(0) \sin \beta, \quad (2)$$

$$(a\lambda + b)y(1) = (c\lambda + d)y'(1), \quad (3)$$

where $a, b, c, d \in \mathbb{R}$, $0 \leq \beta < \pi$, $ad - bc < 0$, and $q \in C[0, 1]$ is real-valued. Suppose that $y(x, \lambda)$ is the solution of (1) with initial conditions

$$y'(0) = \cos \beta, \quad y(0) = \sin \beta. \quad (4)$$

The characteristic function is defined as

$$\omega(\lambda) = (a\lambda + b)y(1, \lambda) - (c\lambda + d)y'(1, \lambda). \quad (5)$$

For convenience, we introduce the following notation: Let

$$\tilde{y}_1 = \lim_{\lambda \rightarrow \lambda_0} y_\lambda(x, \lambda),$$

and $y_1 = \tilde{y}_1 + Cy_0$ for some constant C . If λ_0 is an eigenvalue of multiplicity two, then define $y_1^* = y_1 + C_1y_0$, where

$$C_1 = -\frac{\mathfrak{A}(\hat{y}_1)\frac{\omega''(\lambda_0)}{2} + \mathfrak{A}(y_0)\frac{\omega'''(\lambda_0)}{6}}{\mathfrak{A}(y_0)\frac{\omega''(\lambda_0)}{2}}. \quad (6)$$

$$\mathfrak{A}(y_1^*) = \begin{cases} \frac{y_1^*(1)}{c\lambda_0 + d} - \frac{cy_0(1)}{(c\lambda_0 + d)^2}, & \text{if } \lambda_0 \neq -\frac{d}{c}, \\ \frac{(y_1^*)'(1)}{a\lambda_0 + b} - \frac{ay_0'(1)}{(a\lambda_0 + b)^2}, & \text{if } \lambda_0 = -\frac{d}{c}. \end{cases} \quad (7)$$

If λ_0 is an eigenvalue of multiplicity two, then define $y_1^* = y_1 + C_1y_0$, where

$$C_1 = -\frac{\mathfrak{A}(\hat{y}_1)\frac{\omega''(\lambda_0)}{2} + \mathfrak{A}(y_0)\frac{\omega'''(\lambda_0)}{6}}{\mathfrak{A}(y_0)\frac{\omega''(\lambda_0)}{2}}. \quad (8)$$

If λ_0 is a triple eigenvalue, then define $y_1^\# = y_1 + C_2y_0$, where

$$C_2 = -\frac{\mathfrak{A}(\hat{y}_1)\frac{\omega'''(\lambda_0)}{6} + \mathfrak{A}(y_0)\frac{\omega^{IV}(\lambda_0)}{24}}{\mathfrak{A}(y_0)\frac{\omega'''(\lambda_0)}{6}}. \quad (9)$$

Let

$$\tilde{y}_2 = \frac{1}{2} \lim_{\lambda \rightarrow \lambda_0} y_{\lambda\lambda}(x, \lambda),$$

and $y_2 = \tilde{y}_2 + C\tilde{y}_1 + Dy_0$ for some constant D .

$$\mathfrak{A}(y_1^\#) = \begin{cases} \frac{y_1^\#(1)}{c\lambda_0 + d} - \frac{cy_0(1)}{(c\lambda_0 + d)^2}, & \text{if } \lambda_0 \neq -\frac{d}{c}, \\ \frac{(y_1^\#)'(1)}{a\lambda_0 + b} - \frac{ay_0'(1)}{(a\lambda_0 + b)^2}, & \text{if } \lambda_0 = -\frac{d}{c}. \end{cases} \quad (10)$$

For convenience, we introduce the following notations: Let

$$\tilde{y}_1 = \lim_{\lambda \rightarrow \lambda_0} y_\lambda(x, \lambda),$$

and $y_1 = \tilde{y}_1 + Cy_0$ for some constant C . If λ_0 is an eigenvalue of multiplicity two, then define $y_1^* = y_1 + C_1y_0$, where

$$C_1 = -\frac{\mathfrak{A}(\hat{y}_1)\frac{\omega''(\lambda_0)}{2} + \mathfrak{A}(y_0)\frac{\omega'''(\lambda_0)}{6}}{\mathfrak{A}(y_0)\frac{\omega''(\lambda_0)}{2}}. \quad (11)$$

$$\mathfrak{A}(y_1^*) = \begin{cases} \frac{y_1^*(1)}{c\lambda_0 + d} - \frac{cy_0(1)}{(c\lambda_0 + d)^2}, & \text{if } \lambda_0 \neq -\frac{d}{c}, \\ \frac{(y_1^*)'(1)}{a\lambda_0 + b} - \frac{ay_0'(1)}{(a\lambda_0 + b)^2}, & \text{if } \lambda_0 = -\frac{d}{c}. \end{cases} \quad (12)$$

If λ_0 is a triple eigenvalue, then define $y_1^\# = y_1 + C_2y_0$, where

$$C_2 = -\frac{\mathfrak{A}(\hat{y}_1)\frac{\omega'''(\lambda_0)}{6} + \mathfrak{A}(y_0)\frac{\omega^{IV}(\lambda_0)}{24}}{\mathfrak{A}(y_0)\frac{\omega'''(\lambda_0)}{6}}. \quad (13)$$

$$\mathfrak{A}(y_1^\#) = \begin{cases} \frac{y_1^\#(1)}{c\lambda_0 + d} - \frac{cy_0(1)}{(c\lambda_0 + d)^2}, & \text{if } \lambda_0 \neq -\frac{d}{c}, \\ \frac{(y_1^\#)'(1)}{a\lambda_0 + b} - \frac{ay_0'(1)}{(a\lambda_0 + b)^2}, & \text{if } \lambda_0 = -\frac{d}{c}. \end{cases} \quad (14)$$

If λ_0 is a triple eigenvalue, then define $y_2^\# = y_2^* + D_1y_0$ where

$$D_1 = -\frac{\mathfrak{A}(\hat{y}_2)\frac{\omega'''(\lambda_0)}{6} + \mathfrak{A}(\hat{y}_1)\frac{\omega^{IV}(\lambda_0)}{24} + \mathfrak{A}(y_0)\frac{\omega^V(\lambda_0)}{120} - C_2^2\mathfrak{A}(y_0)\frac{\omega'''(\lambda_0)}{6}}{\mathfrak{A}(y_0)\frac{\omega'''(\lambda_0)}{6}}. \quad (15)$$

$$\mathfrak{A}(y_2^\#) = \begin{cases} \frac{y_2^\#(1)}{c\lambda_0 + d} - \frac{cy_1^\#(1)}{(c\lambda_0 + d)^2} + \frac{c^2y_0(1)}{(c\lambda_0 + d)^3}, & \text{if } \lambda_0 \neq -\frac{d}{c}, \\ \frac{(y_2^\#)'(1)}{a\lambda_0 + b} - \frac{a(y_1^\#)'(1)}{(a\lambda_0 + b)^2} + \frac{a^2y_0'(1)}{(a\lambda_0 + b)^3}, & \text{if } \lambda_0 = -\frac{d}{c}. \end{cases} \quad (16)$$

Theorem 1. If λ_0 is a double eigenvalue, then $\mathfrak{A}(y_1^*) = 0$ if and only if $C = -\frac{\omega'''(\lambda_0)}{3\omega''(\lambda_0)}$.

Theorem 2. If λ_0 is a triple eigenvalue, then $\mathfrak{A}(y_1^\#) = 0$ if and only if $C = -\frac{\omega^{IV}(\lambda_0)}{4\omega'''(\lambda_0)}$.

Theorem 3. If λ_0 is a triple eigenvalue, then $\mathfrak{A}(y_2^\#) = 0$ if and only if

$$D = C^2 + \frac{\omega^{IV}(\lambda_0)}{4\omega'''(\lambda_0)} \left(C + \frac{\omega^{IV}(\lambda_0)}{4\omega'''(\lambda_0)} \right) - \frac{\omega^V(\lambda_0)}{20\omega'''(\lambda_0)} \quad (17)$$

References

- [1] **N. B. Kerimov, Y. N. Aliyev**, *The basis property in L_p of the boundary value problem rationally dependent on the eigenparameter*, *Studia Mathematica* **174**(2) (2006), 201–212.
- [2] **Y. N. Aliyev**, *Minimality of the system of root functions of Sturm-Liouville problems with decreasing affine boundary conditions*, *Colloquium Mathematicum* **109**(1) (2007), 147–162.
- [3] **Y. Aliyev, N. Aliyeva** *Sturm-Liouville problems with a boundary condition depending linearly on an eigenparameter*. Preprint (2026).

On The Inverse Nodal Problem For Diffusion Operator

Rauf Amirov

Institute of Mathematics, Baku, Azerbaijan

(e-mail: amirovrauf@gmail.com)

Consider the following boundary value problem:

$$-y'' + [q(x) + 2\lambda p(x)]y = \lambda^2 y, 0 < x < \pi, \quad (1)$$

$$y'(0) - hy(0) = 0, y'(\pi) + Hy(\pi) = 0, \quad (2)$$

$$h|y(0)|^2 + H|y(\pi)|^2 + \int_0^\pi \left[|y'(x)|^2 + q(x)|y(x)|^2 \right] dx > 0. \quad (3)$$

Here λ is a spectral parameter, the functions $q(x) \in L_2[0, \pi]$ and $p(x) \in W_2^1[0, \pi]$ and $h, H \in \mathbb{R}_+$.

The inverse problems play an important role in mathematics, mathematical physics and have many applications in natural sciences. Inverse spectral problems consist in recovering operators from their spectral characteristics. One of the methods for solving inverse problems for diffusion operators is to utilize the zeros of the eigenfunctions. These zeros are also referred to as nodal points. The endeavor to reconstruct the coefficients of the diffusion operator from the asymptotic formula of the nodal points is known as the inverse nodal problem.

In 1998, the inverse nodal problem was posed and solved for Sturm-Liouville problems by J.R. McLaughlin [1], who showed that the knowledge of a dense subset of nodal points of the eigenfunctions alone can determine the potential function of the Sturm-Liouville problem up to a constant. Recently, some authors have reconstructed the potential function for generalizations of the Sturm-Liouville and Diffusion problem from the nodal points (for example, [2-6]).

In the presented study, necessary and sufficient conditions for solving the inverse nodal problem for Diffusion operator on finite interval have been obtained.

References

- [1] J.R. McLaughlin, Inverse spectral theory using nodal points as data-A uniqueness result, *Jour. of Diff. Eq.*, 1988, 73(2), pp. 354-362.
- [2] O.H. Hald, J.R. McLaughlin, Solutions of inverse nodal problems, *Inverse Problems*, 1989, 5, pp. 307-347.
- [3] C.L. Shen, C.T. Shieh, An inverse nodal problem for vectorial Sturm-Liouville equation, *Inverse Problems*, 2000, 16, pp. 349-356.
- [4] Ch.-F. Yang, Inverse nodal problems of discontinuous Sturm-Liouville operator, *J. Differ. Equ.*, 2013, 254, pp. 1992-2014.
- [5] R.Kh. Amirov, Reconstruction of the Sturm-Liouville operator from nodal data, *Filomat*, 2024, 38,14, pp. 5051-5060.
- [6] Ch.-F. Yang, Reconstruction of the Diffusion operator from nodal data, *Z. Naturforsch*, 2010, 65a, pp. 100-106.

Second-Order Optimality Conditions for a Variational Problem with a Free Right Endpoint and Terminal Cost

Narmina Asgarova

*Azerbaijan State Oil and Industry University (ASOIU),
French-Azerbaijani University (UFAZ), Baku, Azerbaijan
(e-mail: narmina2000@gmail.com)*

We consider a variational calculus problem with one fixed endpoint

$$J(x(\cdot)) = \int_{t_0}^{t_1} L(t, x, \dot{x}) dt + l(x(t_1)) \rightarrow \min; \quad x(t_0) = x_0. \quad (P)$$

Theorem 1. Necessary conditions for a weak extremum. *Let $\hat{x} \in C^2([t_0, t_1], \mathbb{R})$ be a weak local minimizer of problem (P) ($\hat{x} \in \text{wlocmin } P$), and let the integrand L be two times continuously differentiable in a neighborhood of the extended graph $\Gamma_{\hat{x}, \dot{\hat{x}}}$, i.e. $L \in C^2(\mathcal{O}(\Gamma_{\hat{x}, \dot{\hat{x}}}))$, and let $l \in C^2(\mathcal{O}(\hat{x}(t_0), \hat{x}(t_1)))$. Then the following conditions hold on $\hat{x}$:*

a) *the Euler–Lagrange equation:*

$$-\frac{d}{dt} \hat{L}_{\dot{x}} + \hat{L}_x = 0,$$

and the transversality condition:

$$\hat{L}_{\dot{x}}(t_1) = -\hat{l}_{x(t_1)};$$

b) *the Legendre condition;*

c) *if the strengthened Legendre condition holds, then the Jacobi condition holds;*

d) *if the strengthened Legendre and strengthened Jacobi conditions hold, then for any $h_0, h_1 \in \mathbb{R}$ there exists a unique solution of the Jacobi equation such that*

$$h(t_0) = h_0, \quad h(t_1) = h_1,$$

and the following inequality holds:

$$P(h) + Q(h) = h_1 (\hat{L}_{\dot{x}\dot{x}} \dot{h}(t_1) + \hat{L}_{\dot{x}x} h(t_1)) + \hat{l}''(\hat{x}(t_1)) h_1^2 \geq 0 \quad \forall h_1 \in \mathbb{R},$$

where $h(t)$ is the solution of the Jacobi equation with boundary conditions $h(t_0) = 0$, $h(t_1) = h_1$.

Theorem 2. Sufficient conditions for a strong local minimum.

Let $\hat{x} \in C^2([t_0, t_1], \mathbb{R}^n)$ be an admissible extremal of problem (P), and let the integrand $L \in C^3(V \times \mathbb{R}^n)$, where $V \subset \mathbb{R}^{n+1}$ is a neighborhood of the graph $\Gamma_{\hat{x}}$. Assume that the strengthened Legendre and Jacobi conditions hold on $\hat{x}$, and that

$$P(h) + Q(h) = h_1(\hat{L}_{\dot{x}\dot{x}}\dot{h}(t_1) + \hat{L}_{\dot{x}x}h(t_1)) + \hat{l}''(\hat{x}(t_1))h^2(t_1) \geq 0 \quad \forall h_1 \in \mathbb{R},$$

where $h(t)$ is the solution of the Jacobi equation with boundary conditions $h(t_0) = 0$, $h(t_1) = h_1$. Assume also that the integrand L is convex with respect to $\dot{x}$ on V , and that the terminal function l is twice continuously differentiable at the point $(\hat{x}(t_0), \hat{x}(t_1))$. Then $\hat{x} \in \text{strlocmin } P$.

Quadratic functional We consider the following problem on the representation of a quadratic functional, which plays an important role in deriving necessary and sufficient optimality conditions for a problem with one fixed endpoint:

$$K(h) = \int_{t_0}^{t_1} (A\dot{h}^2 + 2Bh\dot{h} + Ch^2) dt = \int_{t_0}^{t_1} \tilde{L}(t, h, \dot{h}) dt.$$

We assume that $A(\cdot), B(\cdot) \in C^1([t_0, t_1])$ and $C(\cdot) \in C([t_0, t_1])$.

Lemma 1. On the representation of a quadratic functional Let the strengthened Legendre and strengthened Jacobi conditions hold for the integrand $\tilde{L}$ of the functional K . Then the functional K admits the representation

$$K(h) = \int_{t_0}^{t_1} (A\dot{h}^2 + 2Bh\dot{h} + Ch^2) dt = \int_{t_0}^{t_1} A\left(\dot{h} - \frac{\dot{u}}{u}h\right)^2 dt + \left(\frac{A\dot{u}}{u} + B\right)h^2 \Big|_{t=t_1},$$

for all $h \in C^1([t_0, t_1])$ satisfying $h(t_0) = 0$, where $u(\cdot)$ is any solution of the Jacobi equation having no zeros on $(t_0, t_1]$.

Theorem 3. On the quadratic functional in a problem with one fixed endpoint. Let $A, B \in C^1[t_0, t_1]$, $C \in C[t_0, t_1]$, and assume that the strengthened Legendre condition for a minimum holds. Then:

a) if an admissible extremal $\hat{x}$ exists, and the strengthened Jacobi condition holds, and

$$(P + Q)(h) = h_1(A\dot{h}(t_1) + Bh(t_1)) + \alpha h_1^2 \geq 0 \quad \forall h_1 \in \mathbb{R},$$

where $h(t)$ is the solution of the Jacobi equation satisfying $h(t_0) = 0$, $h(t_1) = h_1$, then

$$\hat{x} \in \text{absmin } P;$$

b) if the Jacobi condition is not satisfied, or if the strengthened Jacobi condition holds but $(P + Q)(h) \geq 0$ is violated, then

$$S_{\text{absmin}} = -\infty.$$

Moreover, if an admissible extremal $\hat{x}$ exists, then

$$\hat{x} \notin \text{wlocmin } P.$$

References

- [1] V.M. Alekseev, E.M. Galeev, V.M. Tikhomirov, *Problems in Optimization*, Nauka, Moscow, 1984.
- [2] E.M. Galeev, *Optimization Methods*, Baku, 2016.

The Qualitative Property of Solutions Parabolic Non-Linear Equations

Tahir Gadjev^{a,b,c}, Aybeniz Yagnaliyeva^c, Khalil Aliyev^c

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*

^c*Sumgait State University, Sumgait, Azerbaijan*

(e-mail: tqadjiev@gmail.com, Aybeniz.Yaqnaliyeva@sdu.edu.az,
xelil.aliyev@sdu.edu.az)

For smooth solutions without degenerate this type problem for non-linear elliptic and parabolic equations in many works are considered. In the case of linear uniform parabolic equations optimal regularity of the solution is considered in [1]. The removable results for linear equations is considered in [2].

In cylindrical domains $Q_T = \Omega \times (0, T)$ where $\Omega \subset R^n$, $n \geq 2$ is bounded domains, following degenerate non-linear parabolic equation is considered.

$$u_t - \operatorname{div}(\omega(x) |Du|^{p-2} Du) = 0, \quad u(x, t)|_{\Gamma(Q_T)} = h \quad (1)$$

Denotes the parabolic boundary of Q_T by

$$\Gamma(Q_T) = (\overline{\Omega} \times \{0\}) \bigcup (\partial\Omega \times [0, T])$$

Function $\omega(x)$ is a Muckenhoupt function. Let $C_\omega(Q_T)$ be a weighted space, where with norm following

$$\|f\|_{C_\omega(Q_T)} = \sup_{\substack{x_1, x_2 \in Q_T \\ x_1 \neq x_2}} \frac{|f(x_1)\omega(x_1) - f(x_2)\omega(x_2)|}{\|x_1 - x_2\|_a} < \infty, \quad f: Q_T \rightarrow R \quad c$$

where the parabolic metric is defined as

$$\|(x_1, t_1) - (x_2, t_2)\|_a = \max\left(|x_1 - x_2|, |t_1 - t_2|^{\frac{1}{\rho - \alpha(2-\rho)}}\right)$$

We are now ready to state our first result which concerns optimal regularity for solutions to the problem (1)

Theorem 1. Let's consider problem (1) and let $u(x, t)$ solution of this problem. Let $u \in C_\omega(Q_T)$ and $\overline{Q_T} = \Omega \times (0, T)$. Then

$$u(x, t) \in C_\omega(Q_T).$$

We also give removable theorem for solutions. Let $H_\omega^n(E)$ denote ω -Hausdorff measure of set E . Then we prove following removable singularities.

Theorem 2. Let $Q_T \subset R^{n+1}$ be a cylindrical domain and let $E \subset Q_T$ be a closed set. Assume that $u_\omega(x, t)$ is a weak solution to

$$u_t - \operatorname{div}(\omega(x) |Du|^{p-2} Du) = 0 \text{ in } Q_T/E \text{ and that } u \in C_\omega(Q_T)$$

Assume also

$$H_\omega^n(E) = 0.$$

Then the set E is removable, i.e., $u(x, t)$ can be extended to be a weak solution in Q_T .

Theorem 3. Let u solution to problem and $Q' \subset Q$ a bounded space-time cylinder such that $\hat{Q}' \cap \partial Q = \emptyset$. Then $u \in \hat{C}_{\omega(x)}(Q')$ and

$$|u|_{\hat{C}_{\omega(x)}(Q')} = c(n, p, w(\cdot), \omega(x), Q, Q', \operatorname{osch})$$

Although Theorem 3 concerns optimal internal regularity, we also establish optimal regularity prior to the initial state. In particular, in this case we prove $\hat{C}_{\omega(x)}$ - estimates for $Q' = O' \times (0, T)$, for each $O' \subset O$. Note that in this case Q' is not a compact subset of Q . The result in this case is as follows.

Theorem 4. Let u solution of problem $h(\cdot, 0) \in C_{\omega(x)}(O)$ and $O' \subseteq O, Q' = O' \times (0, T)$. Then

$$u \in \hat{C}_{\omega(x)}(Q')$$

and

$$|u|_{\hat{C}_{\omega(x)}(Q')} = c(n, p, \omega(x), O, O', \operatorname{osch}, [h(\cdot, 0)]_{C_{\omega(x)}(O)})$$

Note that Theorem 3 implies the following result.

Applying Theorem 3, we establish precise conditions for the removability of sets for weak solutions. We will consider the problem in the cylinders introduced in the previous section.

Let d fixed, $0 < d < r_0$ and $E \subset R^{n+1}$.

$L(d, w(\cdot); E) = \{Q_{r_i}(x_i, t_i)\}$ a family of cylindrinids such as, what $E \subseteq \cup Q_{r_i}(x_i, t_i)$ and $0 < r_i < d$ for $i = 1, 2, \dots$.

Using these notations, we introduce the Hausdorff measure

$$H(E) = \lim_{\delta \rightarrow 0} \inf_{L(d, w(\cdot); E)} \left\{ \sum r_i^n : E \subseteq \cup Q_{r_i}(x_i, t_i) \right\}$$

where the infimum is taken with respect to all possible coverings $L(d, E)$ sets E .

Theorem 5 (removable singularities). Let $Q \subset R^{n+1}$ cylindrical region and $E \subset Q$ closed set.

Let's assume that u weak solution

$$u_t - \operatorname{div}(\omega(x)|Du|^{p-2}Du) = 0 \text{ on } Q \setminus E$$

So, that $u \in \hat{C}_{\omega(x), \text{loc}}(Q)$.

Let us assume that $H(E) = 0$. Then the set E removable, i.e. u can be extended to a weak solution in Q .

Note that in the definition of Hausdorff measure, different Hausdorff measures (and dimensions) correspond to different values of functions and metrics for the Hausdorff measure bases. The peculiarity, in our case, is that each time we define, we consider the metric or, equivalently, the cylinders used for coverings and the measurable functions that are related to each other. For a closer comparison with the situation when the standard parabolic Hausdorff measure is used, we note that in the case of $w(r) = r^a$, $a \in (0, 1]$, we obtain

$$H_a^s(E) = \lim_{\delta \rightarrow 0} \inf_{L(\delta, r^a, E)} \left\{ \sum r_i^\sigma : E \subseteq \cup Q_{r_i}^{r_i^a}(x_i, t_i) \right\}$$

In this case, the Lebesgue measure of the cylinder is $Q_{r_i}^{r_i^a}(x_i, t_i)$ used in coating E essentially equals $r_i^{n+a(2-p)+p}$ and by the assumption of Theorem 5 for the set $H_a^{n+a}(E) = 0$. When $a = 1$ the standard parabolic Hausdorff measure is considered.

$$H^\sigma(E) = \lim_{\delta \rightarrow 0} \inf_{L(\delta, r; E)} \left\{ \sum r_i^\sigma : E \subseteq \cup B(x_i, r_i) \times (t_i - r_i^2, t_i + r_i^2) \right\}$$

In this case, the following theorem is true.

References

- [1] Caffarelli L. The obstacle problem revisited, J. Fourier Anal. Appl., 4 (1998).
- [2] Gadjeiev T.S. On removability of compact of solutions in classes bounded functions, Ukr. Math. J. 8 (2014), 38–44.

Inversion Formula for Solving Parametric Problems With Irregular Boundary Conditions for Variable Coefficients Equations

Elmaga Gasymov, Saddein Efendi

Baku State University, Baku, Azerbaijan

(e-mail: elmagagasymov@gmail.com, efendi_sadeddin@mail.ru)

Find the solution of the equation

$$\sum_{j=0}^2 a_j(x) \frac{d^j y}{dx^j} - \lambda^2 y = \psi(x), \quad x \in (0, 1), \quad (1)$$

satisfying the conditions

$$U_i(y) \equiv \sum_{k=0}^n \sum_{\mu=-m_i}^{m_i} \lambda^\mu \left\{ \sum_{s=0}^q \gamma_{ks}^{i\mu} \frac{d^k y}{dx^k} \Big|_{x=\nu_s} + \int_0^1 \gamma_k^{i\mu}(x) \frac{d^k y}{dx^k} dx \right\} = \gamma_i, \quad i = 1, 2, \quad (2)$$

where $a_j(x)$, $\psi(x)$, $\gamma_k^{i\mu}(x)$ are the known functions, $\gamma_{ks}^{i\mu}$, γ_i , ν_s ($0 = \nu_0 < \nu_1 < \dots < \nu_q = 1$) are the known numbers; n, m_i, q are non-negative integers, λ is a complex parameter.

1⁰. Let $a_j(x) \in C^{m+N+j}([0, 1])$, $j = 0, 1, 2$, $|a_2(x)| > 0$ for $x \in [0, 1]$, N ($N \geq 0$)- be an integer.

2⁰. Let $a_2(x) = |a_2(x)| e^{\sqrt{-1} \arg a_2(x)}$, $|\arg a_2(x)| \leq \frac{\pi}{2} - 4\varphi$ for $x \in [0, 1]$, where φ ($0 < \varphi \leq \frac{\pi}{8}$), be a constant.

Assume $\theta(x) = \frac{1}{\sqrt{a_2(x)}}$,

$$R_\varphi = \left\{ \lambda : |\lambda| \geq R, |\arg \lambda| \leq \frac{\pi}{4} + \varphi \right\}.$$

3⁰. Let the functions $\gamma_k^{i\mu(x)}$ in the segment $[h, 1-h]$ be piecewise continuous and $\gamma_k^{i\mu(x)} \in {}^l([0, h] \cup [1-h, 1])$, where $h(0 < h < 1), l(l \geq 0)$ are some numbers

Expanding the determinant

$$\Delta_0(\lambda) = \begin{vmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) \end{vmatrix},$$

we have

$$\Delta_0(\lambda) = b_M \lambda^M + b_{M-1} \lambda^{M-1} + \dots + b_{M-S} \lambda^{M-S} + O(\lambda^{M-S-1}), \text{ for } \lambda \in R_\varphi, \quad (3)$$

where M ($M = m_1 + 2n + m_2$), S are some integers, $b_M, \dots, b_{M-S}$ are some numbers.

4⁰. Let even if one of the numbers

$$b_M, b_{M-1}, \dots, b_{M-S} \text{ (from (3))} \quad (4)$$

be non-zero.

5⁰. Let if $n \leq 2$, then $\psi(x) \in C([0, 1])$, while if $n > 2$, then $\psi(x) \in C^{n-2}([0, 1])$.

We have

Theorem 1. Under the constraints 1⁰-5⁰ for $\lambda \in R_\varphi$ the problem (1),(2) has a unique solution and this solution is represented by the formula

$$y(x, \lambda) = \delta(x, \lambda, \gamma_1, \gamma_2) + \int_0^1 G(x, \xi, \lambda) \psi(\xi) d\xi + \\ + \sum_{j=0}^{n-2} \int_0^1 G_j(x, \xi, \lambda) \frac{d^j}{d\xi^j} \psi(\xi) d\xi + \sum_{j=0}^{n-2} \sum_{s=0}^q \delta_{js}(x, \lambda) \frac{d^j \psi(\xi)}{d\xi^j} \Big|_{\xi=\nu_s}; \quad (5)$$

where

$$\delta(x, \lambda, \gamma_1, \gamma_2) = \frac{1}{\Delta(\lambda)} \begin{vmatrix} 0 & y_1(x, \lambda) & y_2(x, \lambda) \\ -\gamma_1 & U_1(y_1) & U_1(y_2) \\ -\gamma_2 & U_2(y_1) & U_2(y_2) \end{vmatrix},$$

$$\begin{aligned}
G(x, \xi, \lambda) &= \frac{1}{\Delta(\lambda)} \begin{vmatrix} P(x, \xi, \lambda) & y_1(x, \lambda) & y_2(x, \lambda) \\ g_1(\xi, \lambda) & U_1(y_1) & U_1(y_2) \\ g_2(\xi, \lambda) & U_2(y_1) & U_2(y_2) \end{vmatrix}, \\
G_j(x, \xi, \lambda) &= \frac{1}{\Delta(\lambda)} \begin{vmatrix} 0 & y_1(x, \lambda) & y_2(x, \lambda) \\ \tilde{\beta}_j^{(1)}(\xi, \lambda) & U_1(y_1) & U_1(y_2) \\ \tilde{\beta}_j^{(2)}(\xi, \lambda) & U_2(y_1) & U_2(y_2) \end{vmatrix}, \\
\delta_{js}(x, \lambda) &= \frac{1}{\Delta(\lambda)} \begin{vmatrix} 0 & y_1(x, \lambda) & y_2(x, \lambda) \\ \beta_{js}^{(1)}(\lambda) & U_1(y_1) & U_1(y_2) \\ \beta_{js}^{(2)}(\lambda) & U_2(y_1) & U_2(y_2) \end{vmatrix}. \tag{6}
\end{aligned}$$

We have

Theorem 2. Under the constraints 1^0 and 2^0 if the function $\psi(E)$ is price-wise absolutely continuous in the segment $[0, 1]$ and $|\psi'(x)| \in L_p(0, 1)$, ($p > 1$), then for $0 < x \leq 1$ we have the following inversion formulas

$$\int_L \lambda^k d\lambda \int_0^1 P(x, \xi, \lambda) \psi(\xi) d\xi = \begin{cases} 0 & \text{for } k = 0, \\ -\sqrt{-1} \left(\frac{\pi}{2} + 2\varphi \right) \frac{\psi(x-0) + \psi(x+0)}{2}, & \text{for } k = 1, \end{cases} \tag{7}$$

where L is an infinite smooth line in whose quite far part coincides with continuations of the rays $\arg \lambda = \pm \left(\frac{\pi}{4} + \varphi \right)$ R_φ , and in (7) the line integral L is understood in the sense of principal value

Theorem 3. Under the constraints 1^0 - 4^0 , if $\psi(x)$ is an absolutely integrable function in the segment $[01]$, we have

$$\int_L \lambda^s d\lambda \int_0^1 G_j(x, \xi, \lambda) \psi(\xi) d\xi = 0, \text{ as } 0 < x < 1, j = 0, 1, \dots, n-2; \tag{8}$$

s is any integer.

Theorem 4. Under the constraints 1^0 - 4^0 and for $0 < x < 1$ we have

$$\begin{aligned}
\int_L \lambda^s \delta(x, \lambda, \gamma_1, \gamma_2) d\lambda &= 0, \\
\int_L \lambda^s \delta_{ji}(x, \lambda) d\lambda &= 0, \quad j = \overline{0, n-2}, \quad i = \overline{0, q}, \tag{9}
\end{aligned}$$

s is any integer.

We have

Theorem 5. Under the constraints 1⁰-4⁰ and in the case $n \leq 2$ if the function $\psi(x)$ is absolutely continuous in the segment $[0, 1]$ and $|\psi'(x)| \in L_p(0, 1)$, ($p > 1$), and in the case $n > 2$ constraints 5⁰, is fulfilled, then for $x \in (0, 1)$ we have the following inversion formulas

$$\int_L \lambda^k y(x, \lambda) d\lambda = \begin{cases} 0 & \text{for } k = 0, \\ -\sqrt{-1} \left(\frac{\pi}{2} + 2\varphi \right) \frac{\psi(x-0) + \psi(x+0)}{2}, & \text{for } k = 1, \end{cases}$$

where $y(x, \lambda)$ is from (5).

On Duality of the Lagrange Problem with Discrete Inclusions and Multiple Delays

Gülseren Çiçek

Istanbul University Faculty of Science, Istanbul, Türkiye
(e-mail: cicekgul@istanbul.edu.tr)

In many dynamic systems in nature, the change of a particle depends not only on its current state but also on its past states. This situation is formulated using delayed differential equations and inclusions [1, 2, 3, 4]. Besides, the duality theory is important in the analysis of optimization and variational problems [3, 4]. Our paper deals with the Lagrange problem with the multiple delayed first-order differential inclusions and duality. Using the concept of infimal convolution and convex duality functions, we construct dual problem for the differential inclusion and prove the duality results. It is evident that the Euler–Lagrange type inclusions are "duality relations" for both primary and dual problems.

We consider the following Lagrange problem with differential inclusions involving multiple delays, labelled as (PPD):

$$\begin{aligned} \text{minimize } I[x(\cdot)] &= \int_0^1 g(x(t), t) dt \\ x'(t) &\in F(x(t), x(t - \tau_1), x(t - \tau_2), t), \quad \text{a.e. } t \in [0, 1], \quad (\text{PPD}) \\ x(t) &= v(t), \quad t \in [-\tau_2, 0], \quad 1 > \tau_2 > \tau_1 > 0, \\ x(t) &\in \Phi(t), \quad t \in [0, 1], \\ x(1) &\in M, \end{aligned}$$

where $F : \mathbb{R}^{3n} \times [0, 1] \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, $g(\cdot, t) : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^1 \cup \{+\infty\}$ is a convex proper function that is continuous on x , the target set $M \subset \mathbb{R}^n$ is convex, $\Phi(t) : [0, 1] \rightrightarrows \mathbb{R}^n$ is a given convex set-valued mapping. Function v is a given absolutely continuous function defined on $[-\tau_2, 0)$, where τ_1, τ_2 are different positive real constants that $1 > \tau_2 > \tau_1 > 0$ hold. The integral in cost functional I is understood in means of Lebesgue.

A feasible trajectory $x(\cdot)$ in the problem is taken to be an absolutely continuous function on the time interval $[-\tau_2, 1]$ together with the first order derivative for which $x'(\cdot) \in L_1^n([0, 1])$.

Recall that the Hamiltonian function and Argmaximum set for the set-valued mapping F are defined by $H_F(x, y, z^*) = \sup_z \{\langle z^*, z \rangle : z \in F(x, y)\}$, $z^* \in \mathbb{R}^n$, and $F_A(x, u; v^*) = \{v \in F(x, u) : \langle v^*, v \rangle = H_F(x, u, v^*)\}$, respectively, and the locally adjoint mapping (LAM) $F^*(w^*; x_0) = \{x^* : (x^*, -w^*) \in K_{gph F}^*(x_0)\}$. The conjugate function for the Hamiltonian function, taken with a minus sign, $M_F(x^*, y^*, z^*) = \inf_{x, y, z} \{\langle x^*, x \rangle + \langle y^*, y \rangle - H_F(x, y, z^*)\}$.

Theorem 1. Suppose $\tilde{x}(t)$, $t \in [-\tau_2, 1]$, is an admissible trajectory for the convex problem (PPD) and there exist absolutely continuous functions $x^*(t), \xi^*(t), \mu^*(t), x^*$ that satisfy the Euler-Lagrange and transversality inclusions

- (i) $\left(-x^{*'}(t) - \xi^*(t + \tau_1) - \mu^*(t + \tau_2), \xi^*(t), \mu^*(t) \right) \in F^*\left(x^*(t); (\tilde{x}(t), \tilde{x}(t - \tau_1), \tilde{x}(t - \tau_2), \tilde{x}'(t)), t\right) - \{\partial_x g(\tilde{x}(t), t) - K_{\Phi(t)}^*(\tilde{x}(t))\} \times \{0\} \times \{0\},$
a.e. $t \in [0, 1 - \tau_2)$,
- (ii) $\left(-x^{*'}(t) - \xi^*(t + \tau_1), \xi^*(t), \mu^*(t) \right) \in F^*\left(x^*(t); (\tilde{x}(t), \tilde{x}(t - \tau_1), \tilde{x}(t - \tau_2), \tilde{x}'(t)), t\right) - \{\partial_x g(\tilde{x}(t), t) - K_{\Phi(t)}^*(\tilde{x}(t))\} \times \{0\} \times \{0\},$
a.e. $t \in [1 - \tau_2, 1 - \tau_1)$,
- (iii) $\left(-x^{*'}(t), \xi^*(t), \mu^*(t) \right) \in F^*\left(x^*(t); (\tilde{x}(t), \tilde{x}(t - \tau_1), \tilde{x}(t - \tau_2), \tilde{x}'(t)), t\right) - \{\partial_x g(\tilde{x}(t), t) - K_{\Phi(t)}^*(\tilde{x}(t))\} \times \{0\} \times \{0\},$ a.e. $t \in [1 - \tau_1, 1]$,
- (iv) $-x^* + x^*(1) \in K_{\Phi(1)}^*(\tilde{x}(1)), x^* \in K_M^*(\tilde{x}(1)), \xi^*(1) = 0, \mu^*(1) = 0,$
respectively, and condition that ensures the non-emptiness of the LAM F^*
- (v) $\tilde{x}'(t) \in F_A(\tilde{x}(t), \tilde{x}(t - \tau_1), \tilde{x}(t - \tau_2), x^*(t), t),$ a.e. $t \in [0, 1]$.

Then the trajectory $\tilde{x}(t)$ is optimal.

The proof of the theorem is analogous to the proof of Theorem 5.1[2] and we omit it.

Later on $g^*(\cdot, t)$ denotes the conjugate function of the function $g(\cdot, t)$.

According to the convex duality, the following convex problem, denoted (DPD), is the dual to the primary problem (PPD):

$$\sup_{\substack{x^*, u^*, \xi^*, \mu^* \\ u^*(0)=\xi^*(1)=\mu^*(1)=0}} I_*(x^*(t), u^*(t), \xi^*(t), \mu^*(t)) \quad (\text{DPD})$$

where $I_* = -\int_0^1 g^*(u^*(t), t)dt + \langle x(0), x^*(0) \rangle$
 $-\int_0^{1-\tau_2} M_F(-x^{*'}(t) - \xi^*(t + \tau_1) - \mu^*(t + \tau_2) - \phi^*(t) - u^*(t), \xi^*(t), \mu^*(t))dt$
 $-\int_{1-\tau_2}^{1-\tau_1} M_F(-x^{*'}(t) - \xi^*(t + \tau_1) - \phi^*(t) - u^*(t), \xi^*(t), \mu^*(t))dt$
 $-\int_{1-\tau_1}^1 M_F(-x^{*'}(t) - \phi^*(t) - u^*(t), \xi^*(t), \mu^*(t))dt + \int_{-\tau_2}^0 \langle v(t), \mu^*(t + \tau_2) \rangle dt$
 $+ \int_{-\tau_1}^0 \langle v(t), \xi^*(t + \tau_1) \rangle dt - \int_0^1 W_{\Phi(t)}(-\phi^*(t))dt - W_M(-\phi^*(1) + x^*(1)),$
the function $v(t), t \in [-\tau_2, 0]$ is the initial function, $x^*(t), \xi^*(t), \mu^*(t), t \in [0, 1]$ are absolutely continuous functions satisfying Lipschitz condition and $u^*(t)$ is a summable function.

Theorem 2. Let g be a continuous and proper, convex function, $F(\cdot, t)$ be a convex set-valued mapping and $\Phi(t), t \in [0, 1], M$ be convex sets. Moreover, let $\tilde{x}(t)$ be an optimal solution of the primary problem (PPD). Then, for the optimality of the functions $\{x^*(t), u^*(t), \xi^*(t), \mu^*(t)\}$ in the dual problem (DPD), it is necessary and sufficient that the conditions (i)–(v) of the Theorem 1 are satisfied. Besides, the optimal values in the primary (PPD) and dual (DPD) problems are equal.

References

- [1] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag: Berlin, 1984.
- [2] G. Çiçek, *Optimization of discrete and differential inclusions of Lagrange Problem involving different delays*, Istanbul J. of Math, 2025, 3(2), 48–60, DOI: 10.26650/ijmath.2025.00027.
- [3] E.N. Mahmudov, *Approximation and Optimization of Discrete and Differential Inclusions*, Elsevier, Boston, 2011.

- [4] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications*, Springer, Berlin, 2006.

Optimal Control Problem for Second Order Hyperbolic Equations With Control in Coefficients

Hamlet Guliyev^a, Tunzale Huseynova^b

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan State Pedagogical University, Baku, Azerbaijan*
(e-mail: hamletguliyev51@gmail.com, htunzale.bsu@mail.ru)

Let the process be described by a system of differential equations

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(v(x) \frac{\partial u}{\partial x_i} \right) = f(x, t), \quad (1)$$

in the cylinder $Q = \Omega \times (0, T)$, where $u = u(x, t)$ is a state function of the system, $f(x, t) \in L_2(Q)$, here Ω -bounded region in R^n with smooth boundry $\partial\Omega$.

As the class of admissible controls the set $V = \left\{ v(x) : v(x) \in W_2^1[\Omega], 0 < \nu \leq v(x) \leq \mu, \left| \frac{dv(x)}{dx_i} \right| \leq M \right\}$, is taken, here ν , μ and M - are given constant number.

Suppose that

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = \varphi_1(x), \quad (2)$$

$$\frac{\partial u}{\partial \nu} \Big|_S = 0 \quad (3)$$

where ν -the exterior normal of lateral surface $S = \partial\Omega \times (0, T)$ of cylinder Q . We will assume that, the data of the problem satisfy conditions $\varphi_0(x) \in W_2^1(\Omega)$, $\varphi_1(x) \in L_2(\Omega)$ are given vector functions.

If the function $v(x)$ is given, then it is easy to prove that [6] problem (1)-(3) has a unique generalized solution from $W_2^1(Q)$.

If $v(x)$ is an unknown function, then to determine $v(x)$ we set an additional condition

$$u(x, T) = \chi(x), \quad (4)$$

where $\chi(x) \in L_2(Q)$ given function.

We reduce this problem to the following optimal control problem: find a function $v(x) \in V$, that minimizes the functional

$$J(v) = \frac{1}{2} \int_{\Omega} [u(x, T) - \chi(x)]^2 dx + \frac{\alpha}{2} \|v\|_{W_2^1(\Omega)}^2 \quad (5)$$

together with the solution of the boundary value problem (1)-(3).

In the this work, the optimal control problem for the second order hyperbolic equation with the control in coefficient is studied. Existence theorem proved and necessary condition for optimality was obtained in the form of an integral inequality.

References

- [1] S.I.Kabanikhin, Inverse and ill-posed problems. *Novosibirsk: Siberian Scientific Publishing House*, 2009, 457 p.
- [2] H.F.Guliev, V.N.Nasibzade. Inverse problem of determining the right side of the equation of string vibrations, *Bulletin of Baku University, Series of Physical and Mathematical Sciences*, 2016, No. 2, 19-28
- [3] H.F.Guliev, G.G.Ismailova, On the determination of the right-hand side of the string vibration equation with nonlocal boundary conditions. *News of baku university, Physico-mathematical sciences*, 2015, no. 3, 27-33
- [4] H.F.Guliev, Kh.I.Seyfullayeva, Determination the right hand side of the linear equation of oscillations of plate-like constructions. *Advanced Mathematical Models and Applications* 10(1), 2025, 17-25.
- [5] I. R.Valitov, Inverse problems for hyperbolic equations: a case of unknown factors, time-dependent. *Abstracts of International Conference "Tikhonov and contemporary mathematics"*. Moscow, 2006. 208 p.
- [6] O.A. Ladyzhenskaya, Boundary value problems of mathematical physics. Moscow: Nauka, 1973, 408 p.

Numerical Solution of the Problem of Determining the Initial Function In a Problem for a String Vibration Equation

Hamlet Guliyev^a, Vusala Nasibzadeh^b

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Sumgayit State University, Sumgayit, Azerbaijan*

(e-mail: hamletguliyev51@gmail.com, vusala.nasibzada@sdu.edu.az)

This work is dedicated to determining the numerical solution of the string oscillation equation for the mixed problem in the initial function. This problem reduces to the problem of minimizing a functional constructed using additional information. The problem is solved numerically using the gradient projection method. Not that in [2] some related problems were considered.

1. Problem statement. Let us consider a problem of finding the pair $(u(x, t), v(x))$ from the following relations:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + a_0(x, t)u + f(x, t); \quad (x, t) \in Q \equiv (0, l) \times (0, T), \quad (1)$$

$$u(x, 0) = u_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = v(x), \quad 0 \leq x \leq l, \quad (2)$$

$$u(x, 0) = 0, \quad u(l, t) = 0, \quad 0 \leq t \leq T, \quad (3)$$

$$u(x, T) = \varphi(x), \quad 0 \leq x \leq l, \quad (4)$$

here $u_0(x, t)$, $f(x, t)$, $u_0(x)$, $\varphi(x)$ are the given functions.

Let us reduce this inverse problem to the following optimal control problem:

$$L_\beta(v) = \frac{1}{2} \int_0^l [u(u, T; v) - \varphi(x)]^2 dx + \frac{\beta}{2} \int_0^l [v(x) - \omega(x)]^2 dx \quad (5)$$

Find the minimum

$$v(x) \in V = \left\{ v(x) / \int_0^l v^2(x) dx \leq r^2 \right\} \quad (6)$$

or

$$v(x) \in V = \{v(x) / |v(x)| \leq M\} \quad (7)$$

of this functional within the constraints (1)-(3) here,

$\omega(x)$ is a given function r, M are the given positive numbers. It is considered that all the data of the problem are rather regular.

Applying the gradient projection method, the problem is solved numerically. If the set is in the form of (6), then in the problem (1)-(3), (5), (6) the gradient projection method is reduced to constructing the minimizing sequence $\{v^2(x)\}$ as follows :

$$v_{k+1}(x) = \begin{cases} v_k(x) - \alpha_k J'_\beta(v_k), \\ \int_0^\ell [v_k(x) - \alpha_k J'_\beta(v_k)]^2 dx \leq r^2 \text{ for,} \\ r(v_k(x) - \alpha_k J'_\beta(v_k)) \\ \left(\int_0^\ell [v_k(x) - \alpha_k J'_\beta(v_k)]^2 dx \right)^{\frac{1}{2}} \\ \int_0^\ell [v_k(x) - \alpha_k J'_\beta(v_k)]^2 dx > r^2 \text{ for,} \end{cases},$$

here, $\alpha_k > 0$ parameter is selected by one of the methods described in [1], $J'_\beta(v)$ is a gradient of the functional (5), and is determined as

$$J'_\beta(v) = -\psi(x, 0; v) + \beta[v(x) - \omega(x)]$$

here, $\psi(x, t; v)$ is the solutions of the problem

$$\begin{aligned} \frac{\partial^2 \psi}{\partial t^2} &= \frac{\partial^2 \psi}{\partial x^2} + a_0 \psi, \quad (x, t) \in Q, \\ \psi(x, T) &= 0, \quad \frac{\partial \psi(x, T)}{\partial t} = u(x, T; v) - \varphi(x), \quad 0 \leq x \leq \ell, \\ \psi(0, t) &= \psi(\ell, t) = 0, \quad 0 \leq t \leq T \end{aligned}$$

that is the adjoint of the problem (1)-(3),(5),(6).

If the set V is in the form of (7) the gradient projection method in the problem (1)-(3),(5),(7) is reduced to constructing the minimizing sequence $\{v_k(x)\}$ as follows:

$$v_{k+1}(x) = \begin{cases} -M, & v_k(x) - \alpha_k J'_\beta(v_k) < -M \text{ for} \\ v_k(x) - \alpha_k J'_\beta(v_k), & |v_k(x) - \alpha_k J'_\beta(v_k)| \leq M \text{ for,} \\ M, & v_k(x) - \alpha_k J'_\beta(v_k) > M \text{ for.} \end{cases}$$

Numerical solutions of the problem. For numerical solution of the boundary value problem (1)-(3) in the set $[0, l] \times [0, T]$ we introduce a regular grid domain in the form of

For numerical solution of the boundary value problem (1)-(3) in the set $[0, l] \times [0, T]$ we introduce a regular grid domain in the form of

$$\omega = \{(x_i, t_j) : x_i = ih, t_j = j\tau, i = \overline{0, n_x}, j = \overline{0, n_t}\}, n_x = [l/h], n_t = [T/\tau]$$

here, h, τ are the given positive numbers and are the steps of the grid. Denote by $y_{ij} = u(x_i, t_j)$ the grid function.

Approximate differential equations when the initial and boundary conditions are $i = 1, \dots, n_x - 1, j = 1, \dots, n_t$:

$$\frac{y_i^{j+1} - 2y_i^j + y_i^{j-1}}{\tau^2} = \frac{y_{i+1}^{j+1} - 2y_i^{j+1} + y_{i-1}^{j+1}}{h^2} - a_{0i}^{j+1} y_i^{j+1} + f_i^j, \quad (8)$$

$$y_i^0 = u_{0i}, y_i^1 = u_{0i} + \tau v_i + 0.5\tau^2 [u''_0(x_i) + f_i^0 - v_i u'_0(x_i)], i = 0, \dots, n_x, \quad (9)$$

$$y_0^j = y_1^j, y_n^j = y_{n-1}^j, \quad (10)$$

here $u_{0i} = u_0(x_i)$, $a_{0i}^j = a_1(x_i, t_j)$, $f_i^j = f(x_i, t_j)$, $v_i = v(x_i)$, $i = \overline{0, n_x}$, $j = \overline{0, n_t}$ are grid functions. It is known that the approximation error of the difference boundary value problem is as $O(\tau^2)$. After simple transformations, instead of (8) we obtain the following system of there-diagonal algebraic equations:

$$A_i y_{i+1}^{j+1} - C_i y_i^{j+1} + B_i y_{i-1}^{j+1} = -F_i^j \quad (11)$$

Here we accepted the denotations

$$A_i = B_i = \frac{\tau^2}{h^2} C_i = 1 + 2\frac{\tau^2}{h^2} + \tau^2 a_{0i}^j, F_i^j = 2y_i^j - y_i^{j-1} + \tau^2 f_i^j$$

References

- [1] Vasil'ev, F.P. Methods for solving extremum problems Moscow: Nauka , 1981. 400 p.
- [2] Guliyev H.F., Seyfullayeva Kh.I. Optimal Control Problem for a Nonhomogeneous Equation of Vibrations of a Three-Layer Plate, WSEAS Transactions on Systems and Control, Vol.21, No.1, 2026, pp.122-129,

An Optimal Control Problem for Unstable Parabolic Equation With a Nonlocal Boundary Condition

Hamlet Guliyev^{a,b}, Hikmat Tagiyev^{b,c,d}

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Baku State University, Baku, Azerbaijan*

^c*Azerbaijan Academy of Public Administration under the President of the Republic of Azerbaijan, Baku, Azerbaijan,*

^d*Sumgait State University, Sumgait Azerbaijan*

(e-mail:hamletguliyev51@gmail.com, tagiyevht@gmail.com)

In practice, a number of mathematical physics problems are described by partial differential equations with nonlocal boundary conditions [1]. Optimal control problems for processes described by these equations were considered in [2,4]. In addition, optimal control problems for parabolic equations with nonlocal boundary condition problem also attract attention.

Optimal control problems for stable partial differential equations is studied in [3]. In the presented work, an optimal control problem for unstable parabolic equation with nonlocal boundary conditions is studied. The existence theorem is proved and a necessary condition for optimality is obtained in the form of an integral inequality

Problem statement. Assume that in the cylinder $Q = \Omega \times (0, T)$, $\Omega \subset R^n$, $n = 2$ or 3 , the control process is described by the parabolic equation

$$\frac{\partial u}{\partial t} - Au - u^3 = v(x, t), (x, t) \in Q = \Omega \times (0, T), \quad (1)$$

with initial and integral condition

$$u(x, 0) = \varphi(x), x \in \Omega, \quad \left. \frac{\partial u}{\partial \nu_A} \right|_S = \int_{\Omega} K(x, y) u(y, t) dy, (x, t) \in S, \quad (2)$$

here $Au = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right)$, $\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial u}{\partial x_i} \cos(\nu, x_i)$, ν -outward normal, $a_{ij}(x, t) \in L_{\infty}(Q)$, $\forall \xi \in R^n$, $\sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geq \alpha \sum_{i,j=1}^n \xi_i^2$, $\alpha = \text{const} >$

0 for almost everywhere $(x, t) \in Q$, $\max_Q \left| \frac{\partial a_{ij}(x, t)}{\partial x_k} \right| < \infty$, $u(x, t)$ is the state of the system, $v(x, t)$ is a control function, $\Omega \subset R^n$ is a bounded domain Ω , $S = \partial\Omega \times (0, T)$ - Q is the lateral side of the cylinder, $\partial\Omega$ is the smooth boundary of the domain Ω , $\varphi(x) \in W_2^1(O)$, $K(x, y) \in L_\infty(\Omega \times \Omega)$, $K(x, y) = K(y, x)$, $\frac{\partial K(x, y)}{\partial x} \in L_2(\Omega \times \Omega)$ are the given functions.

The pair (v, u) is said to be an admissible pair, if $v(x, t) \in V$, V close, convex set in $L_2(Q)$ and $V \neq \emptyset$, $u(x, t) \in L_6(Q)$ and almost everywhere satisfies (1)-(2).

It is assumed that the set of admissible pairs is not empty (3)

Suppose that in the set of admissible controls it is required to find the minimum of the functional

$$J(v, u) = \frac{1}{6} \|u - u_d\|_{L_6(Q)}^6 + \frac{N}{2} \|v\|_{L_2(Q)}^2, \quad (4)$$

here $u_d \in L_6(Q)$ is a given function, N is a given positive number.

Theorem 1. Provided the above conditions on the data of the problem (1)-(4), the considered problem has an optimal pair.

Convergence of the adapted penalty method. We introduce the following functional:

$$\begin{aligned} J_\varepsilon^a(v, u) = & \frac{1}{6} \|u - u_d\|_{L_6(Q)}^6 + \frac{N}{2} \|v\|_{L_2(Q)}^2 + \frac{1}{2\varepsilon} \left\| \frac{\partial u}{\partial t} - Au - u^3 - v \right\|_{L_2(Q)}^2 + \\ & + \frac{1}{2} \|u - u_0\|_{L_2(Q)}^2 + \frac{1}{2} \|v - v_0\|_{L_2(Q)}^2, \end{aligned} \quad (5)$$

here

$$\begin{aligned} v \in V, \quad u \in L_6(Q), \quad \frac{\partial u}{\partial t} - Au \in L_2(Q), \\ u(x, 0) = \varphi(x), \quad \frac{\partial u}{\partial \nu_A} \Big|_S = \int_\Omega K(x, y) u(y, t) dy, \quad (x, t) \in S, \end{aligned} \quad (6)$$

$\varepsilon > 0$ is a penalty parameter. The functional (5) is said to be a functional adapted to the pair $\{v_0, u_0\}$.

Assume that the pair $\{v_\varepsilon, u_\varepsilon\}$ is the solution of the problem

$$J_\varepsilon^a(v_\varepsilon, u_\varepsilon) \rightarrow \inf J_\varepsilon^a(v, u), \quad (7)$$

so that $\{v, u\}$ satisfies the conditions (6). We can prove the existence of such a pair as in theorem 1.

Theorem 2. Assume that $\{v_0, u_0\}$ is an optimal pair in the problem (1)-(4). Then there exists such a function $\psi(x, t)$ that the following relations are valid :

$$\frac{\partial u_0}{\partial t} - Au_0 - u_0^3 = v_0(x, t), (x, t) \in Q = \Omega \times (0, T),$$

$$u_0(x, 0) = \varphi(x), x \in \Omega, \quad \left. \frac{\partial u_0}{\partial \nu_A} \right|_S = \int_{\Omega} K(x, y) u_0(y, t) dy, (x, t) \in S,$$

$$-\frac{\partial \psi}{\partial t} - A\psi - 3u_0^2\psi - \int_{\partial\Omega} K(\xi, x)\psi(\xi, t)d\xi = (u_0 - u_d)^5, \quad (8)$$

$$\psi(x, T) = 0, x \in \Omega, \quad \left. \frac{\partial \psi}{\partial \nu_A} \right|_S = 0, (x, t) \in S, u \in W_2^{2,1}(Q), \quad \psi \in W_2^{2,1,\frac{6}{5}}(Q), \quad (9)$$

$$\int_Q (Nv_0 + \psi)(v - v_0) dxdt \geq 0 \forall v \in V.$$

(8)-(9) is called adjoint problem.

References

- [1] L.S.Pulkina, A.E. Savenkova, A problem with a second kind nonlocal integral condition for an one-dimensional hyperbolic equation. *Vest. Sam. Gos. tech. Univ-ta, ser. phys.-math. Sci.*, 2016, p. 20, No. 2, p. 276-289.
- [2] H.F.Guliyev, H.T.Tagiev, An optimal control problem with non-local conditions for the weakly nonlinear hyperbolic equation //Optimal control applications and methods, 2013, vol. 34, issu 2, p.216-235.
- [3] J.L.Lions, Control of distributed singular systems. Moscow, Nauka, 1987 368 p. (in Russia).
- [4] H.F. Guliyev, Y.S. Gasimov, H.T.Tagiyev, T.M. Huseynova, An the inverse problem of finding the right-hand side of wave equation with nonlocal condition. *Tomsk State University Journal of Mathematics and Mechanics*, 49, (2017), 16-25

On a Problem for Hyperbolic Equations With Discontinuity Conditions

Sevinj Hadiyeva

Sumgayit State University, Sumgayit, Azerbaijan
(e-mail: sevinc.hadiyeva@sdu.edu.az)

Hyperbolic equations with the with discontinuity conditions model physical phenomena (sound, electromagnetic waves, string vibrations) with pulsed or instantaneous effects (see [1]-[4]). Such equations make it possible to study the propagation of waves in media where there are sharp inhomogeneities, and how, for example, a shock wave arises, where the solution may have discontinuities.

Let us introduce the domains $D_1 = \{(x, t) : x > 0, t > x\}$, $D_2 = \{(x, t) : x < 0, t > -x\}$, $D_3 = \{(x, t) : x < 0, x < t < -x\}$ and consider the following problem: Find a function $U(x, t)$ satisfying the equation

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = f(x, t), \quad (x, t) \in D_1 \bigcup D_2 \bigcup D_3, \quad (1)$$

and the conditions

$$U(x, x) = \varphi_1(x), \quad 0 < x < \infty, \quad (2)$$

$$U(x, x) = \varphi_2(x), \quad -\infty < x < 0, \quad (3)$$

$$U(x, -x+0) - U(x, -x-0) = \psi(x), \quad -\infty < x < 0, \quad (4)$$

$$U(-0, t) = \alpha U(+0, t), \quad 0 < t < +\infty, \quad (5)$$

$$U'_x(-0, t) = \beta U'_x(+0, t), \quad 0 < t < +\infty, \quad (6)$$

$$\lim_{x+t \rightarrow +\infty} U(x, t) = 0, \quad (7)$$

where α and β are constants, and $\varphi_1(x)$, $\varphi_2(x)$ and $\psi(x)$ are given functions.

Theorem 1. *Let the function $f(x, t)$ be differentiable and $f(x, t) = 0$ for $t < x$, and for each fixed $x \in (-\infty, \infty)$ the inequality*

$$\int_x^\infty d\tau \int_\tau^{+\infty} |f(\tau, \xi)| d\xi < +\infty,$$

is hold. While the functions $\varphi_1(x)$, $\varphi_2(x)$, $\psi(x)$ are twice differentiable and

$$A\varphi_1(0) = \varphi_2(0), \quad \alpha\varphi_1(0) - \varphi_2(0) = \psi(0), \quad (8)$$

then the solution to problem (1)-(7) can be represented in the form

$$U(x, t) = \varphi_1\left(\frac{x+t}{2}\right) + \frac{1}{2} \int_x^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau, \xi) d\xi, \quad (x, t) \in D_1, \quad (9)$$

$$U(x, t) = A\varphi_1\left(\frac{x+t}{2}\right) + B\varphi_1\left(\frac{t-x}{2}\right) + \frac{1}{2} \int_x^0 d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau, \xi) d\xi + \\ + \frac{A}{2} \int_0^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau, \xi) d\xi + \frac{B}{2} \int_0^{+\infty} d\tau \int_{t-\tau-x}^{t+\tau+x} f(\tau, \xi) d\xi, \quad (x, t) \in D_2, \quad (10)$$

$$U(x, t) = \varphi_2\left(\frac{x+t}{2}\right) + B\varphi_1\left(\frac{t-x}{2}\right) - \psi\left(\frac{t-x}{2}\right) + \frac{1}{2} \int_x^0 d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau, \xi) d\xi + \\ + \frac{A}{2} \int_0^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau, \xi) d\xi + \frac{B}{2} \int_0^{+\infty} d\tau \int_{t-\tau-x}^{t+\tau+x} f(\tau, \xi) d\xi, \quad (x, t) \in D_3, \quad (11)$$

where $A = \frac{1}{2}(\alpha + \beta)$, $B = \frac{1}{2}(\alpha - \beta)$.

Let us outline the proof of the theorem. First of all, as in the work of Levitan [5], we find that for $(x, t) \in D_1$ the solution $U(x, t)$ is represented in the form (9). Now, let $(x, t) \in D_2$. The general solution of equation (1) can be written in the form

$$U(x, t) = \theta_1\left(\frac{t+x}{2}\right) + \theta_2\left(\frac{t-x}{2}\right) + \frac{1}{2} \int_x^0 d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau, \xi) d\xi, \quad (12)$$

where θ_1 and θ_2 are arbitrary twice-differentiable functions. To determine the functions θ_1 and θ_2 , we use conditions (5) and (6) and obtain relation (10). Relation (11) is established analogously.

References

- [1] Albeverio, S., Gestezy F. Hoegh-Krohn R., Holden H. *Some Exactly Solvable Models in Quantum Mechanics*, 1988, Springer, Berlin–New York, 457p.
- [2] Aktosun T. Construction of the half-line potential from the Jost function, *Inverse Problems*, 2004, *20*, 859–876.
- [3] Huseynov H.M., Jamshidipour A.H. On Jost solutions of Shturm-Liouville equations with spectral parameter in discontinuity conditions. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics*, 2010, *30*(4), 61-88.
- [4] Huseynov H.M., Osmanova J.A. On Jost solution of Sturm-Liouville equation with discontinuity conditions . *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics*, 2007, *27*(1), 63-70.
- [5] Levitan B.M. *Inverse Sturm-Liouville Problems*. 1987, De Gruyter, 250 p.

Fractional Optimal Control Analysis of Covid-19 and Dengue Fever Co-Infection Model With Atangana-Baleanu Derivative

Asma Hanif

Government College university, Lahore, Pakistan
(e-mail: asmahanifgcu@gmail.com)

A co-infection with Covid-19 and dengue fever has had worse outcomes due to high mortality rates and longer stays either in isolation or at hospitals. This poses a great threat to a country's economy and overburdened healthcare systems. To effectively deal with these threats, comprehensive approaches to prevent and control Covid-19/dengue fever co-infections are desperately needed. Thus, our focus is to formulate a new co-infection fractional model with the Atangana-Baleanu derivative to explore the dynamics of the deadly combination of Covid-19 and dengue fever and to suggest effective and feasible approaches to restrict the spread of co-infection. In the first part of this paper,

we present Covid-19 and dengue fever sub-models, as well as the co-infection model that is locally asymptotically stable when the respective reproduction numbers are less than unity. We establish the existence and uniqueness results for the solutions of the co-infection model. We extend the model to include a vaccination compartment for the Covid-19 vaccine to susceptible individuals and a treatment compartment to treat dengue-infected individuals as optimal control strategies for disease control. We outline the fundamental requirements for the fractional optimal control problem and illustrate the optimality system for the co-infection model using Pontryagin's principle. We implement the Toufik Atangana approximating scheme to simulate the optimality system to measure the cost-effectiveness of the control strategies. The simulations show the effectiveness of the implemented strategy in determining optimal vaccination and treatment rates that decrease the cost functional to a minimum, thus significantly decreasing the number of infected humans and vectors. Additionally, we visualize a meaningful decrease in infection cases with an increase in the memory index. The findings of this study will provide reasonable disease control suggestions to regions facing Covid-19 and dengue fever co-infection.

References

- [1] D. W. Vaughn, S. Green, S. Kalayanarooj, B. L. Innis, S. Nimmannitya, S. Suntayakorn, et al., Dengue viremia titer, antibody response pattern, and virus serotype correlate with disease severity, *J. Infect. Dis.*, **181** (2000), 2–9.
- [2] C. Li, Y. Lu, J. Liu, X. Wu, Climate change and dengue fever transmission in China: Evidences and challenges, *Sci. Total Environ.*, **622–623** (2018), 493–501.
- [3] A. Abidemi, N. A. B. Aziz, Analysis of deterministic models for dengue disease transmission dynamics with vaccination perspective in Johor, Malaysia, *Int. J. Appl. Comput. Math.*, **8** (2022).
- [4] A. Dwivedi, R. Keval, Analysis for transmission of dengue disease with different class of human population, *Epidemiol. Method.*, **10** (2021).

Some New Results from Combining Taylor Series and Block Pulse Functions

Yousef Jafarzadeh

Department of Mathematics, Ka.C., Islamic Azad University, Karaj, Iran
(e-mail: Yousef.jafarzadeh@iau.ac.ir)

Integro-differential equations (IDEs), particularly higher-order linear Volterra Fredholm integro differential equations, arise frequently in applied mathematics, physics, engineering, and biological modeling. Due to the simultaneous presence of differential and integral terms, obtaining exact analytical solutions for such equations is often difficult or impossible. Consequently, the development of accurate and computationally efficient numerical methods remains an important research topic. In this paper, we introduce a new hybrid numerical method that combines the Taylor series expansion with Block Pulse Functions (BPFs) to approximate the solution of higher-order linear Volterra–Fredholm integro-differential equations. The proposed technique exploits the local approximation power of Taylor polynomials together with the orthogonal and piecewise-constant structure of Block Pulse Functions. By integrating these two approaches, the original integro-differential problem is transformed into a system of linear algebraic equations through a matrix representation. This transformation significantly simplifies the computational procedure and allows the unknown coefficients of the approximate solution to be obtained efficiently using standard numerical algorithms. The theoretical foundation of the method is developed in detail, including derivation of the operational matrices, formulation of the hybrid approximation scheme, and convergence analysis. Several fundamental results and theorems related to the existence, uniqueness, and convergence behavior of the approximate solution are presented. The convergence analysis demonstrates that the proposed method provides accurate approximations under suitable smoothness conditions. Numerical experiments are conducted to illustrate the applicability and effectiveness of the technique. The results show strong agreement between the approximate and exact solutions, confirming that the hybrid Taylor–Block Pulse approach offers improved accuracy and computational feasibility compared with several classical numerical methods.

References

- [1] Y. Jafarzadeh and B. Keramati, Convergence analysis of parabolic basis functions for solving systems of linear and nonlinear Fredholm integral equations *Turk J Math* **41** (2017), 787-796.
- [2] A. Akyuz, M. Sezer, *Chebyshev polynomial solutions of systems of higher-order linear Fredholm-Volterra integro-differential equations. Journal of Franklin Institute* **347** (2005), 688-701.
- [3] Y. Jafarzadeh and B. Keramati, Numerical method for a system of integro-differential equations by Lagrange interpolation. *Asian-European Journal of Mathematics* **29** (2016), 215-232.

The Green Function for a Class of Second Order Parabolic Equations in Paraboloid Type Domains

Nazim Jafarov

Institute of Mathematics, Baku, Azerbaijan
(e-mail: nazim.jafarov.math@gmail.com)

Let E_n be n -dimensional Euclidean space of the points $x = (x_1, \dots, x_n)$, D be bounded domain in E_n with the boundary $\partial D \in C^{1+\lambda}$, $\lambda > 0$, moreover $0 \in D$.

Denote by R_{n+1} – $(n+1)$ -dimensional Euclidean space of the points $(x, t) = (x_1, \dots, x_n, t)$, $R_{n+1}^- = R_{n+1} \cap \{(x, t) : t < 0\}$. The domain $Q \subset R_{n+1}^-$ is said to be a paraboloid-type domain if its section with any hyperplane $t = \tau$ ($\tau < 0$) is of the form $\left\{x : \frac{x}{2\sqrt{-\tau}} \in D\right\}$.

Let $Q_T = Q \cap \{(x, t) : -T < t < 0\}$, $S_T = \partial Q \cap \{(x, t) : -T < t < 0\}$, $D_T = Q \cap \{(x, t) : t = -T\}$, $Q_{T_1, T_2} = Q \cap \{(x, t) : -T_1 < t < -T_2\}$, $T_1 > T_2 > 0$.

In Q_T we consider the following parabolic operator

$$L = \Delta + \mu \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\frac{x_i x_j}{4(-t)} \frac{\partial}{\partial x_j} \right) - \frac{\partial}{\partial t}$$

here Δ is n -dimensional Laplace operator, $-\frac{1}{d^2} < \mu < \infty$, where $d = \sup_{y \in D} |y|$.

We consider the following eigen-value problem:

$$\left. \begin{aligned} \sum_{i,j=1}^n \frac{\partial}{\partial \xi_i} (B(\xi) a_{ij}(\xi) v_{\xi_i}) + \lambda B(\xi) v &= 0, \quad \xi \in D \\ v|_{\partial D} &= 0 \end{aligned} \right\} \quad (1)$$

where $\xi = (\xi_1, \dots, \xi_n)$, $\xi_i = \frac{x}{2\sqrt{-t}}$, $i = 1, \dots, n$, $a_{ij}(\xi) = \delta_{ij} + \mu \xi_i \xi_j$, $i, j = \overline{1, n}$, $B(\xi) = (1 + \mu |\xi|^2)^{-\frac{1}{\mu}}$, λ is a spectral parameter.

Theorem. Let $0 < T_2 < T_1 < T$, $0 < T_4 < T_3 < T$ and $T_4 \geq (1 + \beta)T_1$, where β is a positive constant. Then, if $(y, \tau) \in Q_{T_1, T_2}$, $(x, t) \in Q_{T_3, T_4}$ then for the Green function $P(x, t, y, \tau) \leq C_1 |t|_1^{-\frac{n}{2} - \frac{\lambda_1}{4}} |\tau|^{\frac{\lambda_1}{4}}$, where λ_1 positive eigen value of the eigen-value problem (1), the constant depends only on n, μ, β and domain D .

References

- [1] Yu. A. Alkhutov, I.T.Mamedov, The first boundary value problem for second order nondivergent parabolic equations with discontinuous coefficients, Matem Sb., 1314, (1986) 477-500 .
- [2] Yu. A. Alkhutov, Behavior of solutions of second order parabolic equations in noncylindrical domains, Dokl, Ran 3455, (1995) 583-585.

Optimal Control Problem for Matrix Factorizations of the Helmholtz Equation on the Plane

Davron Juraev^{a,b}, Gulsum Aghayeva^{c,d}

^a*Institute of Military Aviation of the University of Military Security and Defense of the Republic of Uzbekistan, Karshi, Uzbekistan*

^b*Institute of Mathematics Uzbekistan Academy of Sciences, Tashkent, Uzbekistan*

^c*Baku State University, Baku, Azerbaijan*

^d*International Engineering and Technology Institute, Hong Kong, China*
(e-mail: juraevdavron12@gmail.com, gulsumm_agayeva@mail.ru)

The Helmholtz equation is a fundamental mathematical model describing stationary wave phenomena in acoustics, electromagnetics, quantum mechanics, and elasticity. Matrix factorization techniques provide efficient tools for decomposing the corresponding differential operators, enabling stable numerical approximations and efficient optimization procedures. This paper discusses an optimal control formulation associated with matrix factorizations of the two-dimensional Helmholtz equation. The mathematical model, optimization framework, and computational aspects are briefly analyzed.

The Helmholtz equation plays an essential role in modeling time-harmonic wave propagation. In two spatial dimensions it is commonly written as

$$-\Delta u - k^2 u = f, \quad (x, y) \in \Omega \subset \mathbb{R}^2, \quad (1)$$

where $u(x, y)$ denotes the wave field, $k > 0$ is the wave number, and f represents an external source. Appropriate boundary conditions complete the mathematical formulation. The governing Helmholtz equation is given by (1).

The numerical treatment of this equation becomes increasingly difficult for large wave numbers because of oscillatory solutions and indefinite system matrices. Matrix factorization methods offer efficient computational strategies by decomposing discretized operators into structured matrix products. Such decompositions simplify the implementation of iterative solvers and naturally integrate with optimization algorithms. These developments motivate the investigation of optimal control problems constrained by matrix-factorized Helmholtz operators [1, 3].

Consider the control variable $q \in U_{ad}$ acting as either a distributed source or a material parameter. The state equation is

$$A(q)u = f, \quad (2)$$

where $A(q)$ denotes the discretized Helmholtz operator after finite difference or finite element approximation. Assume that

$$A(q) = LU, \quad (3)$$

where L and U are lower and upper triangular matrices obtained through matrix factorization. The state equation is formulated in (2), while the matrix factorization is introduced in (3).

The objective is to determine an admissible control minimizing

$$J(u, q) = \frac{1}{2} \|u - u_d\|^2 + \frac{\alpha}{2} \|q\|^2, \quad (4)$$

where u_d is the desired state and $\alpha > 0$ is the regularization parameter. The objective functional is defined by (4).

The optimal control problem is therefore $\min_{u, q} J(u, q)$ subject to $A(q)u = f$.

The existence of an optimal solution follows under standard assumptions of convexity, coercivity, and bounded admissible controls [1].

Introducing the adjoint variable p , the Lagrangian functional is

$$\mathcal{L}(u, q, p) = J(u, q) + p^T (A(q)u - f).$$

The first-order necessary conditions yield

$$A(q)u = f, \quad (5)$$

$$A(q)^T p = u - u_d, \quad (6)$$

$$\alpha q + \frac{\partial A(q)^T}{\partial q} (pu) = 0. \quad (7)$$

The first-order optimality conditions are presented in (5)-(7). The matrix factorization significantly reduces computational complexity because the

forward and adjoint systems are solved using successive triangular substitutions instead of repeated inversion of the original sparse matrix. Consequently, optimization algorithms such as projected gradient methods or sequential quadratic programming become computationally attractive.

For large-scale problems, sparse LU factorization combined with Krylov subspace methods provides an efficient computational framework. The factorized representation improves numerical stability while preserving sparsity patterns. Moreover, preconditioning strategies derived from incomplete matrix factorizations often accelerate convergence of iterative solvers.

The optimization process alternates between solving the state equation, evaluating the adjoint equation, and updating the control variable until convergence. Such algorithms have demonstrated effectiveness in wave propagation, inverse scattering, and acoustic imaging applications [2].

Optimal control problems governed by the Helmholtz equation remain challenging because of the indefinite nature of the underlying operator. Matrix factorization techniques provide an efficient computational mechanism for solving both state and adjoint equations, thereby improving the overall performance of optimization algorithms. Future research may combine hierarchical matrix approximations, domain decomposition, and machine learning techniques to further reduce computational costs for high-frequency wave simulations.

References

- [1] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971.
- [2] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd ed., SIAM, Philadelphia, 2003.
- [3] L. N. Trefethen, D. Bau III, *Numerical Linear Algebra*, SIAM, Philadelphia, 1997.

Direct and Inverse Problems for a Fourth-Order Mixed-Type Equation With Power Degeneration in a Three-Dimensional Domain

Baxtiyar Kadirkulov^a, Dilfuza Uzaqbaeva^b

^a*Alfraganus University, Tashkent, Uzbekistan*

^{a,b}*V.I.Romanovskiy Institute of Mathematics, Tashkent, Uzbekistan*
(e-mail: b.kadirkulov@afu.uz, uzaqbaevadilfuza1606@gmail.com)

Let $D = \{(x, y) | 0 < x < p, 0 < y < q\}$, $\Omega = \{(x, y, t) | (x, y) \in D, -a < t < b\}$ and α, a, b, p, q are given positive real numbers. We introduce the operator

$$Lu = \begin{cases} u_t(x, y, t) + t^\alpha [u_{xxxx}(x, y, t) + u_{yyyy}(x, y, t)], & t > 0, \\ u_{tt}(x, y, t) + u_{xxxx}(x, y, t) + u_{yyyy}(x, y, t), & t < 0. \end{cases}$$

In the domain Ω for equation

$$Lu = F(x, y, t), \quad (1)$$

where

$$F(x, y, t) = \begin{cases} F_1(x, y, t), & t > 0, \\ F_2(x, y, t), & t < 0, \end{cases}$$

let us consider following problems:

Problem 1. Find the function $u(x, y, t)$ from the class

$$u(x, y, t) \in C_{x,y,t}^{2,2,1}(\bar{\Omega}) \cap C_t^2(\Omega_2) \cap C_{x,y}^{4,4}(\Omega_1 \cup \Omega_2), \quad (2)$$

which satisfies equation (1) and conditions

$$u(0, y, t) = u(p, y, t) = u_{xx}(0, y, t) = u_{xx}(p, y, t) = 0, 0 \leq y \leq q, -a \leq t \leq b, \quad (3)$$

$$u(x, 0, t) = u(x, q, t) = u_{yy}(x, 0, t) = u_{yy}(x, q, t) = 0, 0 \leq x \leq p, -a \leq t \leq b, \quad (4)$$

$$u(x, y, -a) = \varphi(x, y), (x, y) \in \bar{D}, \quad (5)$$

where $F(x, y, t)$ is given function, $\Omega_1 = \Omega \cap \{t > 0\}$, $\Omega_2 = \Omega \cap \{t < 0\}$.

Problem 2. Let $F_1(x, y, t) = f_1(x, y)g_1(t)$, and let the function $g_1(t)$ be known. Find functions $u(x, y, t)$ and $f_1(x, y)$, which satisfy conditions (2)-(5) and

$$u(x, y, b) = \psi(x, y), (x, y) \in \overline{D}, \quad (6)$$

where $F_2(x, y, t)$ and $\psi(x, y)$ are given functions.

Problem 3. Let $F_1(x, y, t) = f_1(x, y)g_1(t)$, and let the function $f_1(x, y)$ be known. Find functions $u(x, y, t)$ and $g_1(t)$, which satisfy conditions (2)-(5) and

$$u(x_0, y_0, t) = h(t), 0 \leq t \leq b, \quad (7)$$

where $F_2(x, y, t)$ and $h_1(t)$ are given functions, (x_0, y_0) is given point in D .

In this work, for a degenerate fourth-order mixed-type equation in a three-dimensional domain, direct and inverse problems are solved. By applying the method of spectral analysis, a criterion for the uniqueness of the solution to the direct problem, was established, and the solution is constructed as the sum of a Fourier series. The problem of small denominators arises when justifying the convergence of the series, therefore, estimates for the separation of small denominators from zero and their corresponding asymptotics are established. Two inverse problems are investigated to determine the right side of an equation consisting of the product of two functions, one of which depends on the spatial variables and the other only on the time variable.

References

- [1] K. B. Sabitov, S. N. Sidorov, *Initial-boundary value problem for a three-dimensional parabolic-hyperbolic type equation*, Differential Equations, 2021, vol. 57, No. 8, pp. 1071–1080
- [2] S.N. Sidorov, *Inverse problems for a three-dimensional equation of parabolic-hyperbolic type in finding time-dependent factors of the right-hand sides*, Lobachevskii Journal of Mathematics, 2022, Vol. 43, No. 12, pp. 3641–3657. DOI: 10.1134/S1995080222150288.
- [3] S.N. Sidorov. *Three-dimensional Initial-boundary Value Problem for a Parabolic-hyperbolic Equation With a Degenerate Parabolic Part*. Azerbaijan Journal of Mathematics V. 12, No 1, 2022, ISSN 2218-6816.

Boundary Value Problems for Equations Not Resolved With Respect to the Highest Derivative

Adil Kahramanov

Baku State University, Baku, Azerbaijan
(e-mail: adilqakhramanov@gmail.com)

The paper considers a boundary value problem for differential equations not resolved with respect to the highest derivative. The problem is considered in the half-space $t > 0, x \in R^n$, for regular equations. In the boundary value problem under consideration, the highest symbol $\overline{p_0}(\xi)$ can vanish and this set defines a real surface of n or fewer dimensions. Note that S.L. Sobolev was the first to consider the Cauchy problem for systems of equations not resolved with respect to the highest derivatives of the unknown functions with respect to t . Note that the boundary conditions at $t=0$ have a general form and contain derivatives with respect to both t and $x \in R^n$. In the paper, algebraic conditions for the solvability of the problem under study are derived. Correct conditions for the solvability of the problem under study are given. Note that the problem is studied in the class of solutions $U(x, t)$ growing in the space H no faster than $t^h, e^{\gamma t} h > 0, \gamma > 0$.

Keywords: Green's function, spaces H and $\mathcal{H}$, algebraic solvability conditions, boundary value problem, Cauchy problem.

Let a boundary value problem be given:

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{i\partial x}\right) u(x, t) = f(x, t), \quad t > 0, \quad x \in R^n \quad (1)$$

with general boundary conditions of the form

$$U_i\left(\frac{\partial}{\partial t}, \frac{\partial}{i\partial x}\right) u(x, t)|_{t=0} = 0, \quad i = \overline{1, k}, \quad (2)$$

where the polynomial $p(z, \xi)$ has orders n and m in z and ξ , respectively, $k = n$ is determined by the following regularity condition: the roots $Z_i(\xi)$ ($i = \overline{1, n}$) of the equation $P(z, \xi) = 0$ for $\xi \in R^n$ split into two groups $\operatorname{Re} Z_i(\xi) = 0$ ($j = \overline{1, k}$), and for the second group $\operatorname{Re} Z_l(\xi) > 0, l = \overline{k+1, n}$.

Let the polynomial $P(z, \xi)$ have the form:

$$P(z, \xi) = \sum_{j=0}^n p_j(\xi) z^{n-j}, \quad p_0(\xi) = [\overline{p_0}(\xi)]^l p_{0,0}(\xi) \quad (3)$$

The equation $\overline{p_0}(\xi) = 0$ defines a real surface of n -th or fewer dimensions, $p_{0,0}(\xi)$ does not vanish for real ξ , l is a natural number.

Theorem. *Let the determinant of the system of equations $p(\xi)$ be nonzero for real ξ . The function $f(x, t)$ belongs to the space $\mathcal{H}$ and grows there no faster than a power of t . Each of the integrals*

$$\int_0^\infty e^{\tau} z^j (\xi) \tau^i |\overline{f}(\xi, t+\tau)| d\tau, \quad j = \overline{k+1, n}; \quad i = \overline{0, n-1};$$

belongs to the space H and grows there no faster than a certain power of t . The function $f(x, t)$ has the form

$$f(x, t) = \left[\overline{p_0} \left(\frac{\partial}{i\partial x_1}, \dots, \frac{\partial}{i\partial x_n} \right) \right]^i B(x, t),$$

where the derivatives are understood in the generalized sense, and the function $B(x, t)$ with respect to x belongs to $L_2(R^n)$.

Then problem (1)-(2) has a unique solution $u(x, t)$, growing no faster than some power of t , and for each $t > 0$ $u(x, t)$ in $\mathcal{H}$ depends on $f(x, t)$ continuously with respect to the spaces H and $\mathcal{H}$ [see [6]].

References

- [1] Sobolev S.L. Bulletin of the USSR Academy of Sciences, Mat. Series, 1954.18.3-50.
- [2] Pavlov A.L. Mat. Collection 103(??), 1977. N3(??), 367-391.
- [3] Gasymov M.G. Bulletin of the Arm. SSR Academy of Sciences, Mat. Series VI. 1971. N2-3, p.137-147.
- [4] Kahramanov A.Sh. Bulletin of the Azerbaijan SSR Academy of Sciences, Phys.-Tech. and Mathematical Sciences Series 1980, N6, p.21-28.

- [5] Zarnitzkaya N.V., Selezneva R.G., Eidelman. Siberian Math. Journal. 1974, 15, N2. p.332-342.
- [6] Shilov G.E. Mat. analysis. Second special course. Moscow, science, 1865.
- [7] Kahramanov A.Sh., Khalilova K.G. Journal of Contemporary Applied Mathematics, Vol. 9, N 1, 2019, July, p. 23-30.
- [8] Kahramanov K.Sh., Kahramanov S. Sh., Abdullaev N.A., Abdullaev Yu.A., Aligulieva H.V., Kahramanov A.Sh. Proceedings of the XY Russian Conference on Semiconductor Physics. "Semiconductors-2022", October 3-7, 2022. N Novgorod.
- [9] Kahramanov A.Sh. On the spectral theory of linear differential operators with discontinuous coefficients. Journal of Contemporary Applied Mathematics, Vol. 4, N1, 2014, p. 23-26.

On one Boundary Value Problem for Singularly Perturbed Quasilinear Differential Equation of Arbitrary Odd Order

Mahbuba Kerimova

Azerbaijan State Oil and Industry University, Baku, Azerbaijan
(e-mail: maxbuba3773@mail.ru)

The concept of onecharacteristic differential equation was introduced by M.I.Vishik and L.A. Lusternik in work [6].

An equation of odd order $2k + 1$ of the form

$$L_{2k+1}u \equiv M_1(M_{2k}u) + N_{2k}u = f$$

is called one characteristic, if M_1 is the first order operator, M_{2k} is elliptic operator of order $2k$, and N_{2k} is any differential operator of order $2k$. Obviously, the real characteristics of this equation are only first order characteristics of operator M_1 .

In work [6] mutual degenerations of one characteristic and elliptic equations were studied. In particular, in the square $D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$ the following boundary value problem was considered:

$$\varepsilon^2 \frac{\partial}{\partial x}(\Delta u) - \varepsilon \Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = f(x, y) \quad (1)$$

$$u|_{\Gamma} = 0, \quad \frac{\partial u}{\partial x} \Big|_{x=1} = 0. \quad (2)$$

Here $\varepsilon > 0$ is a small parameter, Γ is a boundary of the square D , $f(x, y)$ is a given function. Assuming here that function $f(x, y)$ becomes zero of the corresponding order, they only constructed first terms of asymptotics of the solution of the boundary value problem (1),(2).

The authors of work [1], abandoning the condition of vanishing of $f(x, y)$ at $x = y$ constructed the first terms of asymptotics of the solution of the problem (1),(2) taking into account inner layers arising near $x = y$. In the same work the asymptotics of solution of this problem was also constructed up to any degree of small parameter.

In works [2]-[5] boundary value problems for singularly perturbed onecharacteristic equations were also investigated. However, in all the works mentioned, the boundary value problems are considered either in rectangular domains or in semi-infinite and infinite regions.

In the present paper a boundary value problem for the equation (1) is considered in curvilinear trapezoid.

Let $x = \varphi_1(y)$, $x = \varphi_2(y)$ be sufficiently smooth functions defined in $[a, b]$ and satisfying the following conditions:

- 1) $\varphi_1(y) < \varphi_2(y)$ at $y \in [a, b]$;
- 2) $\varphi_1(y) \leq y$, $\varphi_2(y) \geq y$ at $y \in [a, b]$
- 3) $\varphi_1(a) = a$, $\varphi_2(b) = b$;
- 4) $\varphi_1'(y) < 1$, $\varphi_2'(y) < 1$ at $y \in [a, b]$

Examples of such functions include $\varphi_1(y) = y - (y - a)^3$, $\varphi_2(y) = (b - y)^3 + y$, $y \in [a, b]$.

Let's introduce some notations: $\Gamma_1 = \{(x, y) | x = \varphi_1(y), a \leq y \leq b\}$, $\Gamma_2 = \{(x, y) | \varphi_1(y) \leq x \leq \varphi_2(y), y = b\}$,

$\Gamma_3 = \{(x, y) | x = \varphi_2(y), a \leq y \leq b\}$, $\Gamma_4 = \{(x, y) | \varphi_1(y) \leq x \leq \varphi_2(y), y = a\}$, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

In $\Omega = \{(x, y) | \varphi_1(y) \leq x \leq \varphi_2(y), a \leq y \leq b\}$ the boundary value problem for the equation (1) is considered under the following boundary conditions:

$$u|_{\Gamma} = 0, \quad \frac{\partial u}{\partial x} \Big|_{\Gamma_3} = 0. \quad (3)$$

Our goal is to construct the asymptotic expansion of the solution of boundary value problem (1), (3) with respect to the small parameter. Due to this iterative processes were carried out, and as a result, the following theorem was proved.

Theorem. Let the function $f(x, y) \in C^{2n+3}(\Omega)$, satisfy the conditions

$$\begin{aligned} \frac{\partial^i f(t, x)}{\partial x^{i_1} \partial y^{i_2}} \Big|_{t=x} &= 0; \quad \frac{\partial^i f(\varphi_2(y), a)}{\partial x^{i_1} \partial y^{i_2}} = 0; \quad \frac{\partial^i f(\varphi_1(y), a)}{\partial x^{i_1} \partial y^{i_2}} = 0; \\ i &= i_1 + i_2; \quad i = 0, 1, \dots, 2n + 3. \end{aligned}$$

Then for the solution of the problem (1),(3) the following asymptotic representation holds:

$$u = \sum_{i=0}^n \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + \sum_{j=0}^{n+1} \varepsilon^j \eta_j + \varepsilon^{n+1} z.$$

Here the functions W_i are defined by the first iterative process, V_j, η_j are boundary layer type functions, close to boundaries Γ_3, Γ_2 which are defined by the other iterative processes, $\varepsilon^{n+1} z$ is remainder term, moreover for the function z the following estimation holds:

$$\varepsilon^2 \left\| \frac{\partial z}{\partial x} \Big|_{x=\varphi_1(y)} \right\|_{L_2(a,b)}^2 + \varepsilon \left[\left\| \frac{\partial z}{\partial x} \right\|_{L_2(\Omega)}^2 + \left\| \frac{\partial z}{\partial y} \right\|_{L_2(\Omega)}^2 \right] + c_1 \|z\|_{L_2(\Omega)}^2 \leq c_2,$$

where $c_1 > 0, c_2 > 0$ are constants, not depending on ε .

References

- [1] M.G. Javadov, M.M. Sabzaliev, On a boundary value problem for an one-characteristic equation degenerating into one-characteristic one, DAN SSSR, 247 (1979), no. 5, 1041-1046.

- [2] Y.Sh.Salimov, M.M.Sabzaliev, Asyptotics of the solution of the boundary value problem for nonclassical type quasilinear equation degenerating into hyperbolic equation, Doklady Akad.Nauk. Rossii, 247(5), 2009, 597-600.
- [3] M.M. Sabzaliev, M.E. Kerimova, Asymptotics of the solution of a boundary value problem for one-characteristic differential equation degenerating into a parabolic equation in an infinite strip, Nonl. Analysis and Differential Equations, 2 (2014), no. 3, 125-133.
- [4] M.M. Sabzaliev, M.E. Kerimova, Asymptotics of the solution in a rectangle of a boundary value problem for one-characteristic differential equation degenerating into a parabolic equation. Transactions of NAS of Azerbaijan, iss. Math. Mech. 34(4), 2014, 97-106
- [5] M.I.Vishik, L.A. Lyusternik: Regular degeneration and boundary layer for linear differential equations with a small parameter, Uspekhi Math. Nauk. 12 5(77), 3-122 (1957).

Conditional Well-Posedness of an Initial-Boundary Value Problem for a Composite Pseudohyperbolic Equation

Ikrombek Khajiev^{a,b}, Elbek Shobdarov^a

^a*National University of Uzbekistan, Tashkent, Uzbekistan*

^b*Turin Polytechnic University in Tashkent, Tashkent, Uzbekistan*
(e-mail: kh.ikrom04@gmail.com; e.b.shobdarov@gmail.com)

In this paper, we investigate an initial-boundary value problem for a composite pseudohyperbolic equation with discontinuous coefficients.

We consider the differential equation

$$\frac{\partial}{\partial t} (u_t(x, t) - \operatorname{sgn}(x)u_{xx}(x, t)) - \operatorname{sgn}(x)u_{xx}(x, t) = f(x, t), \quad (1)$$

in the domain $\Omega = \{(x, t) : -1 < x < 1, x \neq 0, 0 < t < T\}$.

Problem. Find a function $u(x, t)$ in the domain Ω satisfying the equation (1) together with the initial

$$u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x), -1 \leq x \leq 1, \quad (2)$$

boundary

$$u|_{x=-1} = u|_{x=1} = 0, \quad 0 \leq t \leq T, \quad (3)$$

and gluing

$$u|_{x=-0} = u|_{x=+0}, u_x|_{x=-0} = u_x|_{x=+0}, \quad 0 \leq t \leq T, \quad (4)$$

conditions, where $\varphi(x)$, $\psi(x)$ and $f(x, t)$ are sufficiently smooth given functions.

Pseudohyperbolic equations naturally arise in the mathematical modeling of unsteady flows of viscous gases, convective diffusion of salts in porous media, and the propagation of initial compression disturbances [1].

The theory of pseudohyperbolic equations has been extensively studied because of its applications in mathematical physics and continuum mechanics. Fundamental results on Sobolev-type and pseudohyperbolic equations, including the existence, uniqueness, and stability of solutions, were obtained by R.E. Showalter [1]. Boundary value problems for composite-type equations with discontinuous coefficients and a variable direction of evolution were further investigated by A.I. Kozhanov and his collaborators [2]. The theory of ill-posed problems and conditional well-posedness was established by A.N. Tikhonov and further developed by M.M. Lavrentev and his collaborators [3].

The main contribution of this paper is the investigation of the conditional well-posedness of the problem (1)-(4). An a priori estimate for the solution is established, which serves as the basis for constructing the corresponding correctness set. Within this set, the uniqueness of the solution and its conditional stability with respect to the initial data are proved. Furthermore, a regularized solution of the problem is constructed. An appropriate choice of the regularization parameter is proposed to guarantee the continuous dependence of the regularized solution on the perturbed initial data within the correctness set. These results provide a rigorous theoretical foundation for the analysis and stable approximation of the considered ill-posed problem.

References

- [1] R.E. Showalter, *Hilbert Space Methods for Partial Differential Equations*. Pitman, 1977.
- [2] A.I.Kozhanov, *Boundary-Value Problems for Sobolev-Type Equations with Irreversible Operator Coefficient of the Highest Derivatives*. J Math Sci 260, 307–314 (2022).
- [3] M. M. Lavrent'ev, V. G. Romanov, and S. P. Shishatskii, *Ill-Posed Problems of Mathematical Physics and Analysis*, Providence, RI: American Mathematical Society, 1986.
- [4] S. G. Pyatkov, *Solvability of boundary value problems for a second-order equation of mixed type* [in Russian], in: Nonclassical partial differential equations, Collect. Sci. Works, pp. 77-90, Novosibirsk (1988).

Optimal Control of Parabolic Differential Inclusions Depending on First-Order Partial Derivatives

Elimhan Mahmudov^{a,b}, Gökhan Göksu^c

^a*Azerbaijan National Aviation Academy, Baku, Azerbaijan*

^b*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*

^c*Istanbul Technical University, Department of Mathematical Engineering, Istanbul, Türkiye*

(e-mail: elimhan22@yahoo.com, goksug@itu.edu.tr)

This work is concerned with sufficient optimality conditions for generalized parabolic differential inclusions. The essential feature of the model is that the set-valued mapping depends not only on the state variable, but also on the first-order partial derivative of the unknown function. In a rectangular domain $D = [0, L] \times [0, T]$, we consider

$$\min J[u] = \int \int_D f(u(x, t), x, t) dx dt, \quad (1)$$

subject to

$$u_t(x, t) - u_{xx}(x, t) \in F(u(x, t), u_x(x, t), x, t), \quad (x, t) \in D, \quad (2)$$

with prescribed initial and boundary values $u(x, 0) = \alpha_0(x)$, $u(0, t) = \beta_0(t)$ and $u(L, t) = \beta_L(t)$. Here $F : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping and f is a proper convex integrand. This formulation extends earlier parabolic differential inclusion models in which the right-hand side depends only on the unknown function.

The main tool is the locally adjoint mapping (LAM) associated with F , expressed through the Hamiltonian

$$H_F(u_0, u_1, v^*) = \sup\{\langle v, v^* \rangle : v \in F(u_0, u_1)\}. \quad (3)$$

The dependence of F on u_x produces an additional adjoint function. Consequently, the adjoint system is not merely the classical parabolic adjoint equation, but an Euler–Lagrange type differential inclusion with an extra adjoint component.

Theorem 1. *Let $f(\cdot, x, t)$ be a continuous proper convex function and let $F(\cdot, x, t)$ be a convex set-valued mapping. A feasible solution u of (1)–(2) is optimal if there exist classical functions u^* and w^* satisfying*

$$u^*(x, T) = 0, \quad u^*(0, t) = u^*(L, t) = 0, \quad (4)$$

and

$$\begin{aligned} &(-u_t^*(x, t) - u_{xx}^*(x, t) + w_x^*(x, t), w^*(x, t)) \in F^*(u^*(x, t); \\ &(u(x, t), u_x(x, t), u_t(x, t) - u_{xx}(x, t)), x, t) - \partial f(u(x, t), x, t) \times \{0\}, \end{aligned} \quad (5)$$

together with the argmaximum condition

$$u_t(x, t) - u_{xx}(x, t) \in F_A(u(x, t), u_x(x, t); u^*(x, t), x, t). \quad (6)$$

The proof follows from the Hamiltonian inequality generated by the LAM. After comparing u with an arbitrary feasible solution and integrating over D , the terms containing u^* and w^* are transformed into boundary terms. These terms vanish by the initial-boundary conditions of the state and the homogeneous terminal-boundary conditions of the adjoint variable. Hence $J[\tilde{u}] \geq J[u]$ for every feasible $\tilde{u}$.

The result also yields useful special cases. For the linear parabolic control system

$$u_t - u_{xx} = Au + A_1 u_x + Bw, \quad w \in U, \quad (7)$$

the abstract condition reduces to the adjoint equation

$$-u_t^* - u_{xx}^* = -A^* u^* - A_1^* u_x^* - f'_u(u, x, t), \quad (8)$$

with homogeneous terminal-boundary conditions and the maximum condition

$$\langle Bw(x, t), u^*(x, t) \rangle = \max_{v \in U} \langle Bv, u^*(x, t) \rangle. \quad (9)$$

The same method can be applied to nonlinear parabolic equations and to polyhedral parabolic differential inclusions described by finitely many linear inequalities. In the polyhedral case, the sufficient condition takes the form of an adjoint parabolic relation with a nonnegative multiplier, complementarity, and conjugate boundary conditions.

The proposed approach also admits a multidimensional version with the Laplace operator,

$$u_t(x, t) - \Delta u(x, t) \in F(u(x, t), \nabla u(x, t), x, t), \quad (10)$$

where the adjoint inclusion contains the divergence of the auxiliary adjoint vector associated with ∇u . Thus, the framework gives a unified construction of sufficient optimality conditions for parabolic differential inclusions depending on first-order partial derivatives and suggests further extensions to related elliptic and hyperbolic models.

References

- [1] G. Göksu and E.N. Mahmudov, *Time Optimal Control Problem for Second Order LTI Systems*, 8th International Conference on Control and Optimization with Industrial Applications (COIA 2022), Baku, Azerbaijan, 24.08.2022-26.08.2022.
- [2] E.N. Mahmudov, *Approximation and Optimization of Discrete and Differential Inclusions*, Elsevier, Boston, 2011.

- [3] E.N. Mahmudov, *Optimal control of parabolic differential inclusions in one space dimension*, International Journal of Control, 2022, 95(11), 2931–2942.
- [4] E.N. Mahmudov and D. Mastaliyeva, *Optimisation of parabolic type polyhedral discrete, discrete-approximate and differential inclusions*, International Journal of Control, 2024, 97(9), 2073–2083.
- [5] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation II*, Springer, Berlin, 2006.

On the Structure of the Numerical Range of a Multiparameter Problem under the Right Definiteness Condition

Eldar Mammadov

Institute of Mathematics, Baku, Azerbaijan
(e-mail: eldarmuellim@hotmail.com, eldarmuellim@gmail.com)

Consider the multiparameter spectral problem

$$\begin{cases} V_{i0}\varphi_i = \sum_{j=1}^n \lambda_j V_{ij}\varphi_i; & \varphi_i \in H_i \\ i = 1; 2; \dots; n \end{cases} \quad (1)$$

where $V_{i1}; \dots; V_{in}$ are bounded self-adjoint operators in a Hilbert space H_i ; $i = 1; 2; \dots; n$, and V_{i0} are bounded-from-below, self-adjoint operators with compact resolvents, defined on a dense subset in H_i domain $D(T_i) \subset H_i$. The extension of this problem to the tensor product space $H = H_1 \otimes \dots \otimes H_n$ is given by

$$\begin{cases} V_{i0}^t \varphi = \sum_{j=1}^n \lambda_j V_{ij}^t \varphi; & \varphi \in H = H_1 \otimes \dots \otimes H_n \\ i = 1; 2; \dots; n \end{cases} \quad (2)$$

where $V_{ij}^t = I_1 \otimes \dots \otimes I_{i-1} \otimes V_{ij} \otimes I_{i+1} \otimes \dots \otimes I_n$. Separating the spectral parameters in problem (2), we obtain

$$\begin{cases} \Delta_i \varphi = \lambda_i \Delta_0 \varphi_i; & \varphi \in H = H_1 \otimes \dots \otimes H_n \\ i = 1; 2; \dots; n \end{cases} \quad (3)$$

where the operators Δ_i are the operator determinants (algebraic complements) of the elements μ_i in the operator matrix

$$\Delta = \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ V_{10}^t & V_{11}^t & \dots & V_{1n}^t \\ \dots & \dots & \dots & \dots \\ V_{n0}^t & V_{n1}^t & \dots & V_{nn}^t \end{pmatrix}$$

Here, when computing the operator determinant, the product of operators is understood as a tensor product. The vector $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ represents the spectral parameters.

We define the sets

$$M_V = \left\{ (\lambda_1, \dots, \lambda_n) : \sum_{j=1}^n \lambda_j (V_{i,j} \varphi_i, \varphi_i) = (V_{i,0} \varphi_i, \varphi_i), \forall \varphi_i \in H_i, i = 1; \dots; n \right\}, \quad (4)$$

$$M_{V^t} = \left\{ (\lambda_1, \dots, \lambda_n) : \sum_{j=1}^n \lambda_j (V_{i,1}^t \varphi_i, \varphi_i) = (V_{i,0}^t \varphi_i, \varphi_i), \right. \\ \left. \forall \varphi \in D(V_{1,0}) \otimes \dots \otimes D(V_{n,0}), \right\}, \quad (5)$$

$$M_\Delta = \left\{ \vec{\lambda} = (\lambda_1, \dots, \lambda_n) \mid \vec{\lambda} = \left(\frac{(\Delta_1 \varphi, \varphi)}{(\Delta_0 \varphi, \varphi)}, \dots, \frac{(\Delta_n \varphi, \varphi)}{(\Delta_0 \varphi, \varphi)} \right), \right. \\ \left. \forall \varphi \in D(V_{1,0}) \otimes \dots \otimes D(V_{n,0}) \right\} \quad (6)$$

as the **numerical ranges** of problems (1), (2), and (3), respectively.

Assume that these problems satisfy the right-definiteness condition in a non-strict form, i.e.,

$$(\Delta_0 \varphi, \varphi) > 0; \quad \forall \varphi \in H = H_1 \otimes \dots \otimes H_n \quad (7)$$

Generally speaking, the numerical ranges M_V and M_{V^t} are not convex sets, whereas the set M_Δ is convex. The following theorem holds for the numerical ranges M_V , M_{V^t} and M_Δ .

Theorem 1. *If for each fixed parameter value $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ there exists an orthonormal basis $\{e_k\}_{k=1, \dots, \infty}$ in the space H_i and bounded scalar sequences $\{\gamma_{ijk}\}_{k=1, \dots, \infty}$ depending on the parameters $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ such that*

the equalities $\left(\sum_{j=1}^n \lambda_j V_{ij} - V_{i0} \right) e_k = \gamma_{ijk} e_k$, $i, = 1; 2; \dots; n$; $k = 1, \dots, \infty$, hold, then the numerical ranges M_V , M_{V^t} and M_Δ satisfy the following relations:

1. $M_V = M_{V^t} \subseteq M_\Delta$,
2. Additionally, if condition (7) is satisfied, then $\text{conv } M_V = M_\Delta$.

Proof. To prove this theorem, we require the following lemma.

Lemma 1. If (a) a self-adjoint operator A is defined on a dense in the H_1 domain $D(A) \subset H_1$ and (b) there exists an orthonormal basis $\{e_k\}_{k=1, \dots, \infty}$ in the space H_1 and a bounded scalar sequence $\{\gamma_k\}_{k=1, \dots, \infty}$ such that $Ae_k = \gamma_k e_k$, $k = 1, \dots, \infty$, then the numerical ranges M_A and M_{A^t} satisfy the equality $M_A = M_{A^t}$. Here, the operator $A^t = A \otimes I_2$ is defined on the set $H = D(A) \otimes H_2$, where I_2 is the identity operator on the Hilbert space H_2 . The sets

$$M_A = \{ \lambda \mid \lambda = (A\varphi_1, \varphi_1), \forall \varphi_1 \in H_1, \|\varphi_1\| = 1 \},$$

$$M_{A^t} = \{ \lambda \mid \lambda = (A^t\varphi, \varphi), \forall \varphi \in H, \|\varphi\| = 1 \}$$

are the numerical ranges (in the classical sense) of the operators A and A^t , respectively.

Proof. The proof of this lemma follows directly by computing the values of the elements of the corresponding numerical ranges. We also note that this lemma holds for the operators A and $A^t = I_1 \otimes A$.

We now proceed to the proof of Theorem 1. It follows from Lemma 1 that for each fixed parameter value $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$, the numerical ranges (in the classical sense) of the operators $W = \sum_{j=1}^n \lambda_j V_{ij} - V_{i0}$ and $W^t = \sum_{j=1}^n \lambda_j V_{ij}^t - V_{i0}^t$ coincide. This implies that 0 either belongs or does not belong to the classical numerical ranges of the operators W and W^t simultaneously. Consequently, $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ either belongs or does not belong to the sets M_V and M_{V^t} simultaneously, meaning that $M_V = M_{V^t}$ holds. If we consider only decomposable tensors, when computing the elements of the numerical ranges (4) and (6), it is easy to see that the corresponding elements of M_V and M_Δ are equal on these decomposable tensors. Therefore, the inclusion $M_V \subseteq M_\Delta$ holds.

Thus, we have shown that $M_V = M_{V^t} \subseteq M_\Delta$. Let us now prove the second part of the theorem, namely the equality $\text{conv } M_V = M_\Delta$.

Let σ_V , σ_{V^t} and σ_Δ denote the spectra of problems (1), (2), and (3), respectively. It is known that under condition (7) and the aforementioned assumptions, the spectra σ_V , σ_{V^t} and σ_Δ are discrete, real, and $\sigma_V = \sigma_{V^t} = \sigma_\Delta$.

Furthermore, $\text{conv } \sigma_V = \text{conv } M_V$; $\text{conv } \sigma_{V^t} = \text{conv } M_{V^t}$, and $\text{conv } \sigma_\Delta = \text{conv } M_\Delta$. Then, taking into account the convexity of the set M_Δ , we obtain $\text{conv } M_V = M_\Delta$. The theorem is fully proved.

On one Modular Maximin Control Problem

Akmal Mamatov

140104, University Boulevard, 15, Samarkand State University named after
Sh. Rashidov, Samarkand, Uzbekistan
(e-mail: akmalmamatov1964@gmail.com)

Let $\tau = \{\tau_1, \tau_2, \dots, \tau_l\}$ be fixed time instants numbered in ascending order:

$$0 = \tau_1 < \tau_2 < \dots < \tau_l < \tau_{l+1} = t^* < \infty;$$

$$U = \{u(\cdot) | u(t) = \sum_{k=1}^l \bar{u}_k \delta(t - \tau_k), \int_0^{t^*} u(t) dt \leq 1, u(t) \geq 0, t \in T = [0, t^*]\};$$

$$V = \{v(\cdot) | v(t) = \sum_{k=1}^l \bar{v}_k \delta(t - \tau_k), \int_0^{t^*} v(t) dt \leq 1, v(t) \geq 0, t \in T = [0, t^*]\};$$

$$A \in \mathbb{R}^{n \times n}, D \in \mathbb{R}^{n \times r}, b, d \in \mathbb{R}^n, \omega, \omega_*, \omega^* \in \mathbb{R}^r, \Omega = \{\omega \mid \omega_* \leq \omega \leq \omega^*\};$$

h'_i , $i \in I = \{1, 2, \dots, m\}$, be the i -th row of a given constant $m \times n$ matrix H ; g_i , $i \in I$, be the i -th component of a given vector g , $\delta(t)$, $t \in T$, -Dirac delta function. The following problem is considered:

$$J(u(\cdot), \omega) = \min_{v(\cdot) \in V} \sum_{i \in I} |h'_i x(t^*) - g_i| \rightarrow \max_{u(\cdot) \in U, \omega \in \Omega}, \quad (1)$$

$$\dot{x}(t) = Ax(t) + bu(t) + dv(t), t \in T, x(0) = D\omega.$$

For the modular maximin control problem (1), theorems are proven that serve as a basis for developing a solution algorithm for the problem under consideration.

References

- [1] Filippov A.F., *Differential Equations with Discontinuous Right-Hand Sides*, Nauka, Moscow, 1985.

On the Stabilization of Generalized Solution for a Strongly Dissipative Hyperbolic Equation With a Nonlinear Source Term and Nonlinear Boundary Conditions

Elchin Mamedov^{a,b}, Aynur Hasanova^b

^a*Baku State University, Baku, Azerbaijan*

^b*Institute of Mathematics, Baku, Azerbaijan*

(e-mail: elchinmamedov66@gmail.com, aynurhasanova1973@gmail.com)

Nonlinear hyperbolic equations constitute one of the principal research objects in mathematical physics, the theory of elasticity, acoustics, mechanics, and engineering sciences. Such equations are widely used in the mathematical modeling of various physical processes, and the investigation of the long-time behavior of their solutions is regarded as one of the most important directions in the modern theory of partial differential equations [1],[4].

In recent years, considerable attention has been devoted to the study of the influence of boundary dissipation on the asymptotic properties of solutions to hyperbolic equations. In particular, the presence of nonlinear boundary conditions has a significant effect on the time evolution of the energy functional and makes the stabilization problem substantially more complicated. When both internal dissipative terms and nonlinear boundary dissipation are taken into account simultaneously, the investigation of the long-time dynamics of the system becomes of particular interest from both theoretical and applied points of view [4].

On the other hand, the nonlinear function ($f(u)$) appearing on the right-hand side of the equation represents an important mechanism that supplies additional energy to the system or significantly influences its dynamics. In this case, the study of the balance between the nonlinear source and the dissipative

mechanisms becomes one of the central issues in the analysis of the existence, uniqueness, and, in particular, the stabilization of solutions as $t \rightarrow \infty$ ([1],[4]).

Various results concerning problems of this type have been reported in the literature. However, most of them are restricted either to linear boundary conditions, to nonlinearities of particular forms, or to cases involving only internal dissipation or only boundary dissipation. The long-time behavior of solutions under the combined action of strong internal dissipation, nonlinear boundary dissipation, and a nonlinear reaction term remains insufficiently investigated and continues to be an active area of research [4].

In this paper we investigate the stabilization of generalized solutions as $t \rightarrow \infty$ for a nonlinear hyperbolic equation with a strong internal dissipative term and a nonlinear boundary condition. By employing the energy method together with a Lyapunov-type functional, we establish the monotonic decay of the energy functional and derive sufficient conditions characterizing the long-time behavior of generalized solutions ([5].) Furthermore, depending on the properties of the nonlinear boundary dissipation, a general approach is proposed for estimating the decay rate of the energy. This approach constitutes the principal scientific novelty of the present study.

We consider the following initial boundary value problem

$$u_{tt} + \Delta u - \Delta u_t + f(u) = 0, (x, t) \in \Omega \times [0, T] \quad (1)$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega \quad (2)$$

$$u = 0, (x, t) \in S_1 \times [0, T] \quad (3)$$

$$\frac{\partial u}{\partial n} + g(u_t) = 0, (x, t) \in S_2 \times [0, T] \quad (4)$$

where $\Omega \subset R^n$ is bounded domain with smooth boundary $\partial\Omega = S_1 \cup S_2$, $mes S_2 > 0$, where Δ is the Laplace operator, $u_0(x) \in H^1(\Omega)$ is given function, f and g are some nonlinear functions, $\frac{\partial}{\partial n}$ is the external normal vector on $\partial\Omega$. The term Δu_t in the equation characterizes strong material (viscoelastic) dissipation. The function $f(u)$ introduced into the equation describes the internal nonlinearity and is a continuous function satisfying certain structural conditions.

In this paper the stabilization of solutions of the problem (1)-(4) are studied, when nonlinear boundary function has some smoothness properties.

Here we prove the following theorem regarding the stabilization of solutions

Theorem. Let $f : R \rightarrow R$ is local Lipschitz function and

$$F(s) = \int_0^s f(\tau) d\tau \geq 0, \quad (5)$$

$f : R \rightarrow R$ is monotone increasing function and

$$g(s)s \geq c|s|^m, c > 0, m > 1 \quad (6)$$

Then for given initial dates for generalized solution of the problem (1)-(4) the energy function

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \int_{\Omega} F(u) dx$$

is monotone decreasing and

$$\lim_{t \rightarrow \infty} E(t) = E_0 \quad (7)$$

here, $E_0 \geq 0$, is constant.

If, in addition $E_0 = 0$, then

$$\lim_{t \rightarrow \infty} [\|u_t(t)\|_{L_2(\Omega)} + \|\nabla u(t)\|_{L_2(\Omega)}] = 0 \quad (8)$$

References

- [1] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris.,1969.
- [2] H. A. Levine, *Some nonexistence theorems for initial-boundary value problems of nonlinear wave equations*. Archive for Rational Mechanics and Analysis, 1974, 51(5), 371-386.
- [3] V. K., Kalantarov, O. A. Ladyzhenskaya, *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types*. Journal of Soviet Mathematics, L,1978, 10(1), 53-70.
- [4] A. B., Alshin, M. O., Korpusov, & A. G.Sveshnikov, *Blow-up in Non-linear Sobolev Type Equations*. Walter de Gruyter, 2011.

- [5] E. M. Mamedov, On blow up and stabilization of solution for the equation of fourth order with nonlinear boundary conditions, ABSTRACTS Modern Problems of Mathematics and Mechanics of the XII International Scientific Conference Baku, 03-06 September 2025, 284-287

On the Qualitative Properties of Generalized Solutions For The Heat Equation With a Strong Dissipative Term and Nonlinear Boundary Conditions

Elchin Mamedov^{a,b}, Aishan Mammadova^b

^a*Baku State University, Baku, Azerbaijan*

^b*Institute of Mathematics, Baku, Azerbaijan*

(e-mail: elchinmamedov66@gmail.com, aisenmammadova9797@gmail.com)

One of the urgent problems in modern mathematical physics is the qualitative analysis of processes described by nonlinear evolution equations. In particular, strong internal dissipation occurring in a medium, internal nonlinear forces, and nonlinear energy fluxes present on the boundary are widely encountered in various physical, chemical, and biological systems.

We consider the following initial-boundary value problem in a bounded domain $\Omega \subset R^n$, $n \geq 2$ with a smooth boundary $\partial\Omega$:

$$u_t - \Delta u - \Delta u_t + f(u) = 0, (x, t) \in \Omega \times [0, T] \quad (1)$$

$$u(x, 0) = u_0(x), x \in \Omega \quad (2)$$

$$\frac{\partial u}{\partial n} + \frac{\partial u_t}{\partial n} = g(u), x \in \partial\Omega \times [0, T] \quad (3)$$

where Δ is the Laplace operator, $u_0(x) \in H^1(\Omega)$ is given function, $f(u)$ and $g(u)$ are some nonlinear functions, $\frac{\partial}{\partial n}$ is the external normal vector on $\partial\Omega$. The term Δu_t in the equation characterizes strong material (viscoelastic) dissipation. The function $g(u)$ in the boundary condition acts as a nonlinear source entering the system from the outside, which drives the solution to grow towards infinity. The function $f(u)$ introduced into the equation describes the internal nonlinearity and is a continuous function satisfying certain structural conditions [1],[2] .

The simultaneous presence of a strong dissipation operator, along with internal and boundary nonlinearities in the considered model, makes the application of the concept of classical solutions impossible [5]. For this reason, the solution of the problem is considered as a generalized (weak) solution in specialized Sobolev spaces ($H^1(\Omega)$).

Definition. The function $u(x, t)$ belonging to $u(x, t) \in L_\infty(0, T, H^1(\Omega))$ with $u_t \in L_2(0, T, H^1(\Omega))$ on the interval $[0, T]$ is called a general solution of the problem if it satisfies the following integral identity for any test function $v(x, t) \in H^1(\Omega)$:

$$\int_{\Omega} u_t v dx + \int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} \nabla u_t \nabla v dx + \int_{\Omega} f(u) v dx = \int_{\partial\Omega} g(u) v ds \quad (4)$$

As a result of the mathematical analysis, the character of the competition between internal strong dissipation, internal nonlinearity $f(u)$, and the general boundary source $g(u)$ has been fully described, and the following two main theorems have been proved [2],[6].

Theorem 1. (Finite time blow up of the solution):

Assume that for the function $f(u)$ $f(u) \leq \alpha F(u)$, $\alpha > 0$, where $F(u) = \int_0^u f(\tau) d\tau$ and for the function $f(u)$ and the initial function $u_0(x) \in H^1(\Omega)$ the inequality holds

$$E(0) = \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \int_{\Omega} F(u_0) dx - \int_{\partial\Omega} G(u_0) dx \leq 0. \quad (5)$$

Then the generalized solution of the problem (1)-(3) cannot exist globally and blows up in a finite time $T_{\max} < \infty$,

$$\lim_{t \rightarrow T_{\max}} \int_{\Omega} |u(x, t)|^2 dx = \infty \quad (6)$$

Idea of the proof: To prove the theorem, a special functional is constructed, and the nonlinear concavity methodology proposed by H.A. Levine is applied, adapted to the properties of the strong internal dissipation term along with the extended energy methodology developed by V.K. Kalantarov for Sobolev type equations.

Theorem 2. (Global asymptotic stabilization with respect to time): Assume that the internal nonlinearity is absorptive character ($F(u) \geq 0$), the

initial energy is positive ($E(0) > 0$), and the initial data lies inside the potential well determined by the Sobolev embedding constant, i.e.,

$$\|u_0\|_{H^1(\Omega)} \leq \lambda_0 \quad (7)$$

where λ_0 is the positive constant

Then the generalized solution of the problem exists on the global time axis $([0, \infty])$ and stabilizes exponentially towards the trivial stationary solution ($u \equiv 0$), in the norm of the space $H^1(\Omega)$, as $t \rightarrow \infty$, i.e.

$$\|u(\cdot, t)\|_{H^1(\Omega)} \leq C \|u_0\|_{H^1(\Omega)} e^{-\gamma t} \quad (8)$$

where $C > 0$ and $\gamma > 0$ are constants.

Idea of the proof: The proof is achieved using the Lyapunov functional method, utilizing the ability of the strong internal viscosity effect Δu_t to completely damp the energy, alongside differential inequality techniques based on the Kalantarov Levine approaches.

In this study, the qualitative properties of generalized solutions for a parabolic equation with strong internal dissipation, general internal $f(u)$ and general boundary $g(u)$ nonlinearities were fully analyzed. The presented Theorem 1 and Theorem 2 allow predicting the role of internal viscosity, internal nonlinear forces, and boundary effects in real physical processes, as well as determining the long term dynamics and phase portrait of the system.

References

- [1] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris., 1969.
- [2] V. K. Kalantarov, O. A. Ladyzhenskaya, *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types*. Journal of Soviet Mathematics, 1978, 10(1), 53-70.
- [3] J. M Ball, Strongly damped nonlinear wave equations. In *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium 1977*, Vol. 1, pp. 241-256.
- [4] B. A. Bilgin, V.K. Kalantarov, *About blow up of solutions with arbitrary positive initial energy to nonlinear wave equations*. Journal of Mathematical Analysis and Applications, 2015, 426(1), 466-475.

- [5] A. B., Alshin, M. O., Korpusev, & A. G., **Sveshnikov**, *Blow-up in Non-linear Sobolev Type Equations*. Walter de Gruyter., 2011.
- [6] E. M. Mamedov, On Nonlinear problem for the third order equation Karshi Multidisciplinary International Scientific Journal, 2024, 149-154.

The Cauchy Problem for a Class of Nonlinear Dissipative Hyperbolic Equations

Famil Mammadov, Mesud Gahramanov

Nakhchivan State University, Nakhchivan, Azerbaijan

(e-mail: familmemmedov@ndu.edu.az; gehremanovmesud@ndu.edu.az)

In this work, we study the Cauchy problem for a class of nonlinear dissipative hyperbolic equations. Under suitable conditions, the existence and uniqueness of solutions are proved in appropriate functional spaces. We also obtain energy estimates that ensure the stability of solutions and describe their decay in time. The main methods are based on energy techniques, compactness arguments, and Sobolev space tools.

For a quasilinear hyperbolic equation in the domain $[0, \infty) \times \mathbb{R}^n$ with an integral nonlinear anisotropic elliptic term:

$$u_{tt} + u_t + \sum_{i=1}^n (-1)^{l_i} a_i(t, x, [u]_{l,i}) D_{x_i}^{2l_i} u = 0, \quad (1)$$

$$u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x). \quad (2)$$

We consider the Cauchy problem for the following quasilinear hyperbolic equation: $l_1, \dots, l_n \in \{1, 2, \dots\}$. $[u]_{l,i} = \sum_{k=1}^n \beta_{ik} \int_{\mathbb{R}^n} |D^{l_k} u|^2 dx$, $\beta_{ik} \in \mathbb{R}$, $i, k = 1, \dots, n$.

In the special case where $l_1 = \dots = l_n = 1$ and $[u]_{l,i} = \int_{\Omega} |\nabla u|^2 dx$, equation (1) reduces to the well-known dissipative Kirchhoff equation.

Assume that the following conditions are satisfied:

$$a_i(t, \xi) = 1 + a_{1i}(t, \xi), \quad i = 1, \dots, n,$$

where the functions $a_i(t, \xi)$ are differentiable with respect to the variables t and ξ in the domain $[0, \infty) \times (-b, b)$, where $b > 0$. Moreover, $a_{1i}(t, \xi)$ is differentiable with respect to t .

For all $t \in [0, \infty)$, $\xi \in (-b, b)$, the following inequalities hold:

$$|a_{1i}(t, \xi)| \leq c|\xi|^p, \quad |a_{1i,t}(t, \xi)| \leq c|\xi|^p, \quad |a_{1i,\xi}(t, \xi)| \leq c|\xi|^{p-1},$$

where $p > 1$ and $c > 0$ are fixed constants.

We denote

$$H_2 = W_2^{(r+1)l}(\mathbb{R}^n) \times W_2^{rl}(\mathbb{R}^n),$$

and define the inner product in H_r by

$$\langle w^1, w^2 \rangle_{H_r} = \langle u^1, u^2 \rangle_{W_2^{(r+1)l}(\mathbb{R}^n)} + \langle v^1, v^2 \rangle_{W_2^{rl}(\mathbb{R}^n)},$$

where

$$w^1 = \begin{pmatrix} u^1 \\ v^1 \end{pmatrix}, \quad w^2 = \begin{pmatrix} u^2 \\ v^2 \end{pmatrix}, \quad r = 0, 1.$$

We denote the corresponding norm in H_r by

$$\|w\|_{H_r} = \sqrt{\langle w, w \rangle_{H_r}}.$$

Let U_δ^r denote the ball in H_r centered at zero:

$$U_\delta^r = \left\{ w = (v_1, v_2) \in H_r : \|v_1\|_{W_2^{(r+1)l}(\mathbb{R}^n)}^2 + \|v_2\|_{W_2^{rl}(\mathbb{R}^n)}^2 < \delta^2 \right\}.$$

Theorem 1. Assume that conditions (1) and (2) are satisfied. Then there exists a number $\delta_0 > 0$ such that for every $(\varphi, \psi) \in U_{\delta_0}^1$, the Cauchy problem (1), (2) admits a unique solution

$$u \in C([0, \infty); W_2^{2l}(\mathbb{R}^n)) \cap C^1([0, \infty); W_2^l(\mathbb{R}^n)) \cap C^2([0, \infty); L_2(\mathbb{R}^n)).$$

Moreover, the following decay estimates hold:

$$\|D^\alpha u(t, \cdot)\|_{L_2(\mathbb{R}^n)} \leq C_{\delta_0} (1+t)^{-|\alpha/l|}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \cup \{0\}$ and

$$|\alpha/l| = \frac{\alpha_1}{l_1} + \dots + \frac{\alpha_n}{l_n},$$

where $C_{\delta_0} > 0$ is a constant depending on δ_0 .

References

- [1] A.B. Aliyev, N.A. Suleymanov *A mixed problem for some classes quasilinear Sobolev type equation*, Transactions of NAS of Azerbaijan, Issue mathematics and mechanics series of physical-technical and mathematical science, 2004, XXIV(1), 27-37.
- [2] F.V. Mammadov, M.G. Gahramanov, *Cauchy problem for a class of hyperbolic equation with integral nonlinearity and dissipation*, Of the XI International Conference "Modern Problems of Mathematics and Mechanics" dedicated to the memory of a genius Azerbaijani scientist and thinker Nasiraddin Tusi, 03-06 July 2024, 152-153.
- [3] M. Ghisi, M. Gobbino. Global existence and asymptotic behavior for mildly degenerate dissipative hyperbolic equation of kirchhof type // Asymptotic Analysis, 40(2004), p. 25-36.

Sufficient Conditions for a Minimum in a Nonsmooth Delayed Problem of the Calculus of Variations

Samin Malik

ADA University, Baku, Azerbaijan
 Institute of Mathematics, Baku, Azerbaijan
 (e-mail: saminmelik@gmail.com)

In this paper, we investigate a variational problem of the form

$$S(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), x(t-h), \dot{x}(t), \dot{x}(t-h)) dt \rightarrow \min, \quad (1)$$

$$x(t) = \varphi(t), \quad t \in [t_0 - h, t_0], \quad x(t_1) = x_1 \in \mathbb{R}^n. \quad (2)$$

Here $\mathbb{R}^n$ is the n -dimensional Euclidean space, $h = \text{const} > 0$, $t_1 - t_0 > h$, $x(t) \in KC^1([t_0 - h, t_1], \mathbb{R}^n)$, and the function

$$L(t, x, y, \dot{x}, \dot{y}) : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} := (-\infty, +\infty)$$

is a given continuous function and $L(t, x, y, \dot{x}, \dot{y}) = 0$, $\forall t > t_1$. Further, $\varphi(\cdot) \in C^1([t_0 - h, t_0], \mathbb{R}^n)$, where $I := [t_0, t_1]$, and $KC^1([t_0 - h, t_1], \mathbb{R}^n)$ is the set of

piecewise smooth functions, that is, $x(t) \in C([t_0 - h, t_1], \mathbb{R}^n)$ and $\dot{x}(\cdot)$ is a piecewise continuous function on $[t_0 - h, t_1]$.

Let $x(\cdot) \in KC^1([t_0 - h, t_1], \mathbb{R}^n)$ and suppose that it satisfies condition (2). Then the function $x(\cdot)$ is called admissible in problem (1), (2). If an admissible function is a solution of problem (1), (2), then it is a minimum in problem (1), (2). The following theorem is known.

Theorem [1]. Let the function $L(t, x, y, \dot{x}, \dot{y})$ be continuous function with respect to all variables.

Then: (i) if an admissible function $\bar{x}(\cdot)$ is a strong local minimizer in problem (1), (2), then along the function $\bar{x}(\cdot)$ the inequality

$$Q(t, \lambda, \xi; \bar{x}(\cdot)) \geq 0, \quad \forall(t, \lambda, \xi) \in I^* \times [0, 1] \times \mathbb{R}^n \quad (3)$$

holds; (ii) if an admissible function $\bar{x}(\cdot)$ is a weak local minimum in problem (1), (2), then along the function $\bar{x}(\cdot)$ the inequality

$$Q(t, \lambda, \xi; \bar{x}(\cdot)) \geq 0, \quad \forall(t, \lambda, \xi) \in I^* \times [0, 1] \times B_\delta(0) \quad (4)$$

holds. Here $B_\delta(0) \subset \mathbb{R}^n$ is the closed ball with center $0 \in \mathbb{R}^n$ and radius δ . Further, the function $Q(\cdot)$ is defined in work [1].

Definition 1. An admissible function $\bar{x}(\cdot)$ is called an extremal in the strong sense (in the weak sense) if inequality (3) (respectively, (4)) is satisfied along $\bar{x}(\cdot)$.

In various known works, it has been shown that problems described with delay reflect many real processes quite fully. Therefore, the investigation of problem (1), (2) is relevant from both the theoretical and the practical points of view.

Our aim is to obtain sufficient conditions for extremals (in the strong and weak senses) of problem (1), (2). Following the idea proposed in [3], one can obtain a formula for the increment

$$\Delta S(\delta z(\cdot); \bar{z}(\cdot)) = S(\bar{z}(t) + \delta \bar{z}(t)) - S(\bar{z}(t)),$$

where $\bar{z}(t) = (\bar{x}(t), \bar{y}(t))^T$, $\bar{y}(t) = \bar{x}(t - h)$, $\delta z(t) = z(t) - \bar{z}(t)$, and function $\bar{x}(\cdot)$ is an extremal in the weak or local sense in nonsmooth problem (1), (2).

We have:

$$\begin{aligned} \Delta S(\delta z(\cdot); \bar{z}(\cdot)) = & \int_{t_0}^{t_1} \left[L(t, \bar{z}(t), \dot{\bar{z}}(t) + \delta \dot{z}(t)) - \bar{L}(t) \right. \\ & \left. + L(t, \bar{z}(t) + \delta z(t), \dot{\bar{z}}(t) + \delta \dot{z}(t)) - L(t, \bar{z}(t), \dot{\bar{z}}(t) + \delta \dot{z}(t)) \right] dt, \quad (5) \end{aligned}$$

where $\delta z(\cdot) \in KC^1(I, \mathbb{R}^{2n})$ and $\delta z(t_0) = 0$.

Consider the auxiliary set of integrands of the form[arx]

$$\left\{ \widehat{L}(\cdot) = L(\cdot) + \frac{d}{dt} [a^T(t)\delta z(t) + \delta z^T(t)A(t)\delta z(t)], \quad t \in I \right\}, \quad (6)$$

where $\delta z(\cdot)$ and $a(\cdot)$ are $2n$ -dimensional piecewise smooth vector functions such as $a(t_1) = 0$; in addition, $A(\cdot)$ is a $2n \times 2n$ -dimensional piecewise smooth symmetric matrix function such as $A(t_1) = 0$.

By virtue of (6), the increment (5) takes the new form

$$\begin{aligned} \Delta S(\delta z(\cdot); \bar{z}(\cdot)) = & \int_{t_0}^{t_1} \left\{ [L(t, \bar{z}(t), \dot{\bar{z}}(t) + \delta \dot{z}(t)) - \bar{L}(t) + a^T(t)\delta \dot{z}(t)] + \right. \\ & + [L(t, \bar{z}(t) + \delta z(t), \dot{\bar{z}}(t) + \delta \dot{z}(t)) - L(t, \bar{z}(t), \dot{\bar{z}}(t) + \delta \dot{z}(t)) + \\ & \left. + \dot{a}^T(t)\delta z(t) + 2\delta \dot{z}^T(t)A(t)\delta z(t) + \delta z^T(t)\dot{A}(t)\delta z(t)] \right\} dt. \end{aligned} \quad (7)$$

Applying the method of work [2] with the help of the increment formula (7), one can obtain various sufficient conditions for the nonsmooth problem (1), (2). The following theorem is proved.

Theorem 1. Let the integrand $L(t, x, y, \dot{x}, \dot{y})$ be continuous with respect to the collection of variables. Moreover, let an admissible function $\bar{x}(\cdot)$, that is, the function $\bar{z}(t) = (\bar{x}(t), \bar{y}(t))^T$, $\bar{y}(t) = \bar{x}(t - h)$, $t \in I$, be an extremal in problem (1), (2) in the sense of Definition 1. Suppose that there exist a vector function $a(\cdot) \in KC^1(\cdot)$, $a(t_1) = 0$ and a symmetric matrix function $A(\cdot) \in KC^1(\cdot)$, $A(t_1) = 0$, for which, along $\bar{z}(t)$, the inequality

$$\begin{aligned} & L(t, \bar{z}(t), \dot{\bar{z}}(t) + \xi) - \bar{L}(t) + a^T(t)\xi + [L(t, \bar{z}(t) + \eta, \dot{\bar{z}}(t) + \xi) - \\ & - L(t, \bar{z}(t), \dot{\bar{z}}(t) + \xi) + \dot{a}^T(t)\eta + 2\xi^T A(t)\eta + \eta^T \dot{A}(t)\eta] \geq 0, \quad t \in I^*, \end{aligned} \quad (8)$$

$$\forall(\xi, \eta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \quad (\forall(\xi, \eta) \in \mathbb{R}^{2n} \times B_\delta(0) \quad (\forall(\xi, \eta) \in B_\delta(0) \times B_\delta(0))),$$

holds, where $B_\delta(0) \subset \mathbb{R}^{2n}$ is the closed ball with center $0 \in \mathbb{R}^{2n}$ and radius δ . Then: (i) if inequality (8) holds for all $(\xi, \eta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, then a strong local extremal in sense of Definition 1 is an absolute minimum in problem (1),(2);

(ii) if inequality (8) holds for all $(\xi, \eta) \in \mathbb{R}^{2n} \times B_\delta(0)$ ($(\xi, \eta) \in B_\delta(0) \times B_\delta(0)$), then a strong (weak) extremal in the sense of Definition 1 is a strong (weak) local minimum in problem (1),(2).

References

- [1] G.V.Hajiyeva, T.K.Melikov, *Necessary conditions for the extremum in non-smooth problems of variational calculus with delay*, Baku Math J, 2024, 3(2), 136-144.
- [2] M.J.Mardanov, T.K.Melikov, S.T.Malik, *Sufficient conditions for a minimum in nonsmooth problems of the calculus of variations*, Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 2026, 52(1), 183-192.
- [3] M.J. Mardanov, T.K. Melikov, S.T. Malik, *On sufficient conditions in the classical problem of the calculus of variations*, arXiv preprint arXiv:2512.21918, 2025.

Sufficient Conditions for Minimum in Delayed Variational Problems

Misir Mardanov^a, Telman Melikov^b, Aygun Isayeva^{b,c}

^a*Baku State University, Baku, Azerbaijan*

^b*Institute of Mathematics, Baku, Azerbaijan*

^c*Sumgait State University, Sumgait, Azerbaijan*

(e-mail: misirmardanov@yahoo.com, t.melik@rambler.ru,
aygunisayeva32@gmail.com)

In the paper we investigate the variational problem of the form

$$S(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), x(t-h), \dot{x}(t), \dot{x}(t-h)) dt \rightarrow \min, \quad (1)$$

$$x(t) = \varphi(t), \quad t \in [t_0 - h, t_0], \quad x(t_1) = x_1 \in R^n. \quad (2)$$

Here R^n is the n -dimensional Euclidean space, $h = \text{const} > 0$, $t_1 - t_0 > h$, $\varphi(t) \in C^1([t_0 - h, t_0], R^n)$ and the function $L(t, x, y, \dot{x}, \dot{y}) : I \times R^n \times R^n \times R^n \times R^n$ and the functions $L_z(\cdot), L_{\dot{z}}(\cdot), L_{zz}(\cdot), L_{z\dot{z}}(\cdot), L_{\dot{z}\dot{z}}(\cdot)$ are continuous, where $L(t, x, y, \dot{x}, \dot{y}) = 0, \forall t > t_1, \forall (x, y, \dot{x}, \dot{y}) \in R^n \times R^n \times R^n \times R^n, I = [t_0, t_1]$, $z = (x, y)^T, \dot{z} = (\dot{x}, \dot{y})^T$ and the symbol T is the transposition operation.

The function $x(t) \in KC^1(I, R^n)$, where $I := [t_0, t_1]$ and $x(t) = \varphi(t)$, $t \in [t_0 - h, t_0]$, where $KC^1(I, R^n)$ is the set of piecewise smooth functions, i.e. $x(t) \in C(I, R^n)$ and $\dot{x}(t)$ is a piecewise continuous function on I .

Let $x(\cdot) \in KC^1(I, R^n)$ be a function satisfying condition (2). Then function $x(\cdot)$ is called an admissible in problem (1), (2). Let the admissible function be a solution of the problem (1), (2), i.e., it is an extremum in problem (1), (2).

It is important to note that a number of works have shown that problems described with delay (delay in trajectories and in controls), quite fully represent many real processes.

Therefore, the studied problem (1), (2) is relevant from a theoretical as well as a practical point of view.

It is known [2] that if the admissible function $\bar{x}(\cdot)$ is an extremum in problem (1), (2), then the first variation of problem (1), (2) is equal to zero, i.e. the following equality holds

$$\delta^1 S(\delta z(\cdot); \bar{z}(\cdot)) = \int_{t_0}^{t_1} [\bar{L}_z^T(t) \delta z(t) + \bar{L}_{\dot{z}}^T(t) \delta \dot{z}(t)] dt = 0, \quad (3)$$

$$\forall \delta z(\cdot) = (\delta x(\cdot), \delta y(\cdot))^T, \delta y(t) = \delta x(t - h), \delta z(t) = z(t) - \bar{z}(t),$$

$$\bar{z}(t) = (\bar{x}(t), \bar{y}(t))^T, \bar{y}(t) = \bar{x}(t - h), z(t) = (x(t), y(t)), y(t) = x(t - h),$$

where $x(\cdot)$ is arbitrary fixed admissible function, $\bar{L}_\nu^T(t) = L_\nu(t, \bar{z}(t), \dot{\bar{z}}(t))$, $\nu \in \{z, \dot{z}\}$. From (3) we can easily obtain

$$\bar{L}_{\dot{x}}(t) + \bar{L}_{\dot{y}}(t + h) - \int_{t_0}^t [\bar{L}_x(\tau) + \bar{L}_y(\tau + h)] d\tau = c, \quad (4)$$

where $c \in R^n$ is some constant vector, $L(t, x, y, \dot{x}, \dot{y}) = 0$, $\forall t > t_1$.

We also confirm that if the admissible function $\bar{x}(\cdot)$ is a solution of the integral equation (4), then equality (3) is valid along it.

Definition 1. An admissible function $\bar{x}(\cdot)$ is called an extremal in problem (1), (2) if it is a solution of equation (3) or equation (4), i.e. the function $\bar{z}(t) = (\bar{x}(t), \bar{x}(t - h))^T$ satisfies equality (3) or the integral equation (4).

Our goal is to obtain sufficient conditions for the minimality of the extremals $\bar{x}(\cdot)$ of the problem (1), (2).

Let the function $\bar{x}(\cdot)$ be an extremal in problem (1), (2). Then, following the idea proposed in [4] and taking into account (3), we can obtain a formula

for the increment $\Delta S(\delta z(\cdot); \bar{z}(t)) = S(\bar{z}(t) + \delta z(t)) - S(\bar{z}(t))$, where $\bar{z}(t) = (\bar{x}(t), \bar{y}(t))$, $\bar{y}(t) = \bar{x}(t - h)$, $\delta z(t) = z(t) - \bar{z}(t)$.

Using Taylor's formula, we can rewrite it as follows:

$$\begin{aligned} \Delta S(\delta z(\cdot); \bar{z}(t)) = & \int_{t_0}^{t_1} [L(t, \bar{z}(t), \dot{\bar{z}}(t) + \delta \dot{z}(t)) - \bar{L}(t) - \bar{L}_z^T(t) \delta \dot{z}(t) + \\ & + L_z^T(t, \bar{z}(t), \dot{\bar{z}}(t) + \delta \dot{z}(t)) \delta z(t) + \frac{1}{2} \delta z^T(t) L_{zz}(t, \bar{z}(t) + \theta \delta z(t), \dot{\bar{z}}(t) + \\ & + \delta \dot{z}(t)) \delta z(t) - \bar{L}_z^T(t) \delta z(t) + \delta z^T(t) Q(t) \delta \dot{z}(t) + \frac{1}{2} \delta z^T(t) \dot{Q}(t) \delta z(t)] dt, \end{aligned} \quad (5)$$

where $\theta \in (0, 1)$, $Q(t) = Q^T(t)$, $t \in I$, $Q(t_1) = 0$.

Formula (5) allows us to prove the following theorem.

Theorem 1. *Let the admissible function $\bar{x}(\cdot)$ be an extremal in problem (1), (2). Moreover, let there exist a smooth symmetric $n \times n$ matrix $Q(\cdot)$ with the condition $Q(t_1) = 0$. If the inequality*

$$\begin{aligned} & E(L)(t, \bar{z}(t), \dot{\bar{z}}(t) + \xi) + [\xi^T Q(t) + L_z(t, \bar{z}(t), \dot{\bar{z}}(t) + \xi) - \bar{L}_z(t)] \eta + \\ & + \frac{1}{2} \eta^T [L_{zz}(t, \bar{z}(t) + \theta \eta, \dot{\bar{z}}(t) + \xi) + \dot{Q}(t)] \eta \geq 0, \forall t \in I, \forall \theta \in (0, 1), \\ & \forall (\xi, \eta) \in R^{2n} \times R^{2n} (\forall (\xi, \eta) \in R^{2n} \times B_\delta(0) ((\xi, \eta) \in B_\delta(0) \times B_\delta(0))), \end{aligned}$$

holds, then the extremal $\bar{x}(\cdot)$ is an absolute (strong local (weak local)) minimum in problem (1), (2), where $B_\delta(0) \subset R^{2n}$ — a closed ball centered at $O \in R^{2n}$ with radius δ ; $\xi = (p, q)^T \in R^n \times R^n$, $\eta = (l, \mu)^T \in R^n \times R^n$; the function $E(L)(t, z, \dot{z} + \xi)$ is an analogue of the Weierstrass function for problem (1), (2) and it has the form: $E(L)(t, \bar{z}(t), \dot{\bar{z}}(t) + \xi) = L(t, \bar{z}(t), \dot{\bar{z}}(t) + \xi) - \bar{L}(t) - \bar{L}_z(t) \xi$, $t \in I$, $\xi \in R^{2n}$ along $\bar{z}(t) = (\bar{x}(t), \bar{y}(t))^T$.

In this paper, as far as we know, sufficient conditions for a minimum in problem (1), (2), i.e. in a problem with delay, have been obtained for the first time. We also note that, for example, in works [1], [2], [3] only necessary conditions for the minimum were obtained.

References

- [1] G.V.Hajiyeva, T.K.Melikov, *Necessary conditions for the extremum in non-smooth problems of variational calculus with delay*, Baku Math J, 2024, 3(2), 136-144.
- [2] A.M.Isayeva, *Necessary conditions in delayed variational problems with free right end*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, 2022, 42(4), 81-86.
- [3] M.J.Mardanov, T.K.Melikov, G.V.Hajiyeva, *Necessary conditions for a minimum in variational problems with delay in the presence of degeneracies*, Journal of Contemporary Applied Mathematics, 2025, 15(1), 69-91.
- [4] M.J.Mardanov, T.K.Melikov, S.T.Malik, *Sufficient conditions for a minimum in nonsmooth problems of the calculus of variations*, Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 2026, 52(1), 183–192.

Application of the Method of Residues to the Solution of a Mixed Boundary Value Problem for a Class of Variable-Coefficient Differential Equations

Vagif Mastaliyev

Azerbaijan State Pedagogical University, Baku, Azerbaijan
Academy of Public Administration under the President of the Republic of
Azerbaijan, Baku, Azerbaijan
 (e-mail: vagiftrk1@rambler.ru)

The article investigates the existence and uniqueness of the solution to a mixed problem for a class of equations with complex-valued coefficients that behave like parabolic ones, despite the fact that, as "time" progresses, they may transition from parabolic type to Schrödinger type or even to anti-parabolic type.

Problem Statement. This work investigates the solvability of a mixed boundary value problem..

$$M\left(t, \frac{\partial}{\partial t}\right)U = L\left(x, \frac{\partial}{\partial x}\right)U, 0 < t < T, 0 < x < 1 \quad (1)$$

$$U(0, x) = \phi(x) \quad (2)$$

$$l_1(u) = u(t, 0) = 0,$$

$$l_2(u) = u(t, 1) = 0, \quad (3)$$

where $M(t, \frac{\partial}{\partial t}) = P(t) \frac{\partial}{\partial t} + P_0(t)$, $L(x, \frac{\partial}{\partial x}) = \frac{1}{a(x)} \cdot \frac{\partial^2}{\partial x^2}$, $P(t) = \frac{1}{(b-t)(d-t)(c-t)}$, $P_0(t) = p_{01}(t) + ip_{02}(t)$ complex-valued functions, $p_{0j}(t) \in C[0, 1]$, ($j=1, 2$), $b = b_1 + ib_2$, $d = d_1 + id_2$, $c = c_1 + ic_2$ complex numbers, $a(x) > 0$, $a(x) \in C[0, 1]$, $\phi(x)$ – given, a $U(t, x)$ – unknown function.

The final conditions for solvability are given by

$$1^0. \operatorname{Re} \int_0^t \frac{dt}{P(t)} = -\frac{t}{12} (3t^3 + 4Bt^2 + 6Ct + 12D) > 0, \quad 0 < t < T$$

$$B = -\operatorname{Re}(b + c + d), \quad C = \operatorname{Re}(bd + cd + bc), \quad D = -\operatorname{Re}(bcd);$$

$$2^0. a(x) > 0, \quad 0 < x < 1,$$

$$3^0. \phi(x) \in C^2[0, 1], \phi(0) = \phi(1) = 0,$$

$$4^0. [2B^3 - 9BC + 27D]^2 + 4[3C - B^2]^3 < 0, \quad 0 < t_k < T, \quad k = 0, 1, 2,$$

$$t_k = 2\sqrt{-\frac{p}{3}} \cdot \cos\left(\frac{1}{3} \arccos\left(\frac{3q}{2p}\sqrt{-\frac{p}{3}} - \frac{2pk}{3}\right)\right) - \frac{B}{3}, \quad k = 0, 1, 2,$$

$$p = C - \frac{B^2}{3}, \quad q = \frac{2B^3 - 9BC + 27D}{27}.$$

According to the contour integral method scheme, the mixed problem (1)-(3) is associated with the following two auxiliary problems involving a complex parameter λ :

1) A spectral problem consisting of finding the solution to the equation

$$y'' - \lambda^2 a(x) y = -\phi(x), \quad (4)$$

under the boundary conditions

$$l_1(y) \equiv y(0) = 0$$

$$l_2(y) \equiv y(1) = 0 \quad (5)$$

2) The Cauchy problem consists in finding the solution of an equation involving a real parameter and a complex parameter λ .

$$M\left(t, \frac{d}{dt}\right)z - \lambda^2 z = 0, 0 < t < T, \quad (6)$$

under the initial conditions

$$z|_{t=0} = \phi(x), 0 < x < 1 \quad (7)$$

The following theorem holds:

Theorem 1. *Suppose that conditions $1^0, 2^0, 3^0$ and 4^0 hold. If the problem (1)-(3) admits a classical solution in the class $C^{1,2}((0, T] \times [0, 1]) \cap C([0, T] \times [0, 1])$, then this solution is represented in the following form (for $t > 0$):*

$$U(t, x) = - \sum_{k=1}^{\infty} \operatorname{res}_{\lambda_k} \left(e^{\int_0^t \frac{P_0(t) - \lambda^2}{P(t)} dt} \cdot y(x, \lambda) \right) \quad (8)$$

where $y(x, \lambda) = \int_0^1 G(x, \xi, \lambda) \phi(\xi) d\xi$ and $G(x, \xi, \lambda)$ is the Green function of the spectral problem (4)-(5).

References

- [1] Rasulov M.L., *Method of Contour Integral*, Moscow, Nauka – Elm, 1964, 462 pp.
- [2] Rasulov M.L., *Application of the Residue Method to Solving Differential Equation Problems*, Baku – Elm, 1989, 328 pp.
- [3] Naymark M.A., *Linear Differential Operators*, Moscow, Nauka, 1969, 526 pp.
- [4] I.M. Rapoport, *On Some Asymptotic Methods in the Theory of Differential Equations*, Publishing House of the Academy of Sciences of the Ukrainian SSR, Kiev, 1954, 286 pp.
- [5] S.D. Eidelman, *Parabolic Systems*, Moscow, Nauka, 1964, 444 pp.

- [6] Mamedov Yu.A., *Investigation of Correct Solvability of Linear One-Dimensional Mixed Problems for General Systems of Partial Differential Equations with Constant Coefficients*, Baku, 1988. (Preprint / Institute of Physics, Azerbaijan SSR Academy of Sciences, No. 20, 67 pp.)
- [7] Mamedov Yu.A., *On Correct Solvability of General Mixed Problems*, Differential Equations, 1990, Vol. 26, No. 3, pp. 534–537.
- [8] Mamedov Yu.A., Mastaliyev V.Yu., *On the Solvability of Mixed Problems for a New Class of Equations That Can Change from Parabolic to Anti-Parabolic Type*, Vestnik of Baku State University, 2002, No. 4, pp. 93–103.
- [9] Mastaliyev V.Yu., *On Solvability of Mixed Problems for Some Equations Not Classified by Standard Methods*, Proceedings of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan, Vol. XVIII (XXVII), Baku, 2003, pp. 91–96.

First Order Necessary Optimality Conditions in Stochastic Step Control Problem

Rashad Mastaliyev^{a,b}, Nicat Hasanli^c

^a *Azerbaijan University, Baku Azerbaijan*

^b *Institute of Mathematics, Baku, Azerbaijan*

^c *Sumgait State University, Sumgait, Azerbaijan*

(e-mail: rashad.mastaliyev@au.edu.az, nicat.hsnli@mail.ru)

In the abstract, one step optimal control problem described by a system of Ito stochastic differential equations on different time intervals is studied. First order necessary optimality conditions under different conditions were found. In considered problem these necessary conditions appear like stochastic analogs of Pontryagin maximum principle, linearized maximum principle and Euler equation.

Assume that controlled process is described by the following system consisting of the union of ordinary Ito stochastic differential equations and equations with delay:

$$dx(t) = f(t, x(t), x(h(t)), u(t))dt + \\ + \sigma_1(t, x(t), x(h(t)))dw_1(t), \quad t \in [t_0, t_1], \quad (1)$$

$$x(t) = x^0(t), t \in I_{t_0} = [h(t_0), t_0], \\ dy(t) = g(t, y(t), v(t))dt + \sigma_2(t, y(t))dw_2(t), \quad t \in [t_1, t_2], \\ y(t_1) = G(x(t_1)). \quad (2)$$

Here $f(t, x, z, u)(g(t, y, v))$ is given continuous by all arguments $n(m)$ -dimensional vector-function, $z = x(h(t)), x^0(t)$ is given continuous initial function, $\sigma_2(t, x, z)(\sigma_2(t, y))$ is $(n \times n)((m \times m))$ -dimensional matrix-function, $G(x)$ is given m dimensional continuous vector-function, $h(t)$ is given continuous differentiable scalar function, so that $h(t) < t, h'(t) > 0$, $u(t)(v(t))$ are $r(q)$ dimensional partially continuous controlling functions taking values from nonempty bounded $U(V)$ sets, i.e.

$$u(t) \in U \subset R^r, t \in [t_0, t_1], \\ v(t) \in V \subset R^q, t \in [t_1, t_2], \quad (3)$$

All pairs $(u(t), v(t))$ satisfying these conditions will be called possible controls.

Assume that for any possible control there is a unique solution of the problem (1)-(2)

$$S(u, v) = E \{ \varphi_1(x(t_1)) + \varphi_2(y(t_2)) \}. \quad (4)$$

Here E is mathematical expectation of stochastic process, $\varphi_1(x), \varphi_2(y)$ are continuous differentiable scalar functions.

In the paper, first order necessary optimality conditions for considered stochastic step control problem that are stochastic analogs of Pontryagin maximum principle, linearized maximum principle and Euler equation [1-4] are found.

References

- [1] K.B.Mansimov, R.O. Mastaliyev, On the optimality of quasi-singular controls in a stochastic control problem. *Vestnik Tomsk State University. Ser., Control, Computer Engineering and Informatics*, 3(36), 2016, 4-10.

- [2] R.Gabasov , *Singular controls*. Moscow: Nauka, 1979.
- [3] R.O.Mastaliyev, First and second order necessary optimality conditions in control problems described by stochastic differential equations. *J. of Informatics and control problems* 42(2), (2021), 1-15.
- [4] M.J.Mardanov , K.B.Mansimov, T.K. Melikov, *Investigation of sigular controls and second order necessary optimality conditions in systems with delay*, Baku, Elm, 2013, 356 p.

Investigation of Parabolic Fractional Integral Operators (with Rough Kernels) in Parabolic Local Generalized Morrey Spaces

Shemsiyye Muradova

Baku State University, Baku, Azerbaijan
Institute of Mathematics, Baku, Azerbaijan
(e-mail: mshams01@yahoo.com)

Let P be a real $n \times n$ matrix, whose all eigenvalues have positive real part, $A_t = t^P$, $t > 0$, $\gamma = \text{tr} P$ is the homogeneous dimension on R^n and Ω is an A_t -homogeneous of degree zero function, integrable to a power $s > 1$ on the unit sphere generated by corresponding parabolic metric. We find conditions on the pair (φ_1, φ_2) for the boundedness of the operator $I_{\Omega, \alpha}^P f$ from the space $LM_{p, \varphi_1, P}^{\{x_0\}}(R^n)$ to another one $LM_{q, \varphi_2, P}^{\{x_0\}}(R^n)$, $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$, and from the $LM_{1, \varphi_1, P}^{\{x_0\}}(R^n)$ to the weak space $WLM_{p, \varphi_2, P}^{\{x_0\}}(R^n)$, $1 \leq q < \infty$, $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$.

Definition 1. Let $\varphi(x, r)$ be a positive measurable function on $R^n \times (0, \infty)$ and $1 \leq p < \infty$. For any fixed $x_0 \in R^n$ we denote by $LM_{p, \varphi, P}^{\{x_0\}} \equiv LM_{p, \varphi, P}^{\{x_0\}}(R^n)$ the parabolic generalized local Morrey space, the space of all functions $f \in L_p^{loc}(R^n)$ with finite quasinorm

$$\|f\|_{LM_{p, \varphi, P}^{\{x_0\}}} = \|f(x_0 + \cdot)\|_{LM_{p, \varphi, P}}.$$

Also by $WLM_{p,\varphi,P}^{\{x_0\}} \equiv WLM_{p,\varphi,P}^{\{x_0\}}(R^n)$ we denote the weak parabolic generalized local Morrey space of all functions $f \in WL_p^{loc}(R^n)$ for which

$$\|f\|_{WLM_{p,\varphi,P}^{\{x_0\}}} = \|f(x_0 + \cdot)\|_{WLM_{p,\varphi,P}} < \infty.$$

Definition 2. Let $S_\rho = \{w \in R^n : \rho(w) = 1\}$ be the unit ρ -sphere (ellipsoid) in R^n ($n \geq 2$) equipped with the normalized Lebesgue surface measure $d\sigma$ and Ω be A_t -homogeneous of degree zero, i.e. $\Omega(A_tx) \equiv \Omega(x)$, $x \in R^n$, $t > 0$. The parabolic fractional integral operator $I_{\Omega,\alpha}^P f$ with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L_1^{loc}(R^n)$ is defined by

$$I_{\Omega,\alpha}^P f = \int_{R^n} \frac{\Omega(x-y) f(y)}{\rho(x-y)^{\gamma-\alpha}} dy.$$

We prove the boundedness of the parabolic integral operator $I_{\Omega,\alpha}^P$ with rough kernel from one parabolic local generalized Morrey space $LM_{p,\varphi_1,P}^{\{x_0\}}(R^n)$ to another one $LM_{q,\varphi_2,P}^{\{x_0\}}(R^n)$, $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$, and from the space $LM_{1,\varphi_1,P}^{\{x_0\}}(R^n)$ to the weak space $WLM_{q,\varphi_2,P}^{\{x_0\}}(R^n)$, $1 \leq q < \infty$, $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$.

Theorem. Suppose that $x_0 \in R^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t -homogeneous of degree zero. Let $1 \leq p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and the pair (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \frac{\text{ess sup}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{p}}}{t^{\frac{\gamma}{q}+1}} dt \leq C \varphi_2(x_0, r),$$

where C does not depend on x_0 and r . Then the operator $I_{\Omega,\alpha}^P$ is bounded from $LM_{p,\varphi_1,P}^{\{x_0\}}$ to $LM_{q,\varphi_2,P}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1,P}^{\{x_0\}}$ to $WLM_{q,\varphi_2,P}^{\{x_0\}}$ for $p = 1$.

References

- [1] Sh.A. Muradova, A.S. Balakishiyev, N.Z. Orucov. *Parabolic fractional integral operators with rough kernels in parabolic local generalized Morrey spaces*, Caspian Journal of Applied Mathematics, Ecology and Economics, vol.4, no.1, 2016, July, pp. 59-68.

- [2] Sh.A. Muradova, *Parabolic Fractional Integral Operators with Rough Kernels in Parabolic Local Generalized Morrey Spaces*. 6th International HYBRID Conference on Mathematical Advances and Applications, ICOMAA-2023, May 10-13, 2023, Istanbul, TÜRKIYE, p. 239
- [3] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., 43, 1938, pp. 126-166.

Necessary Conditions of Steklov Problem With Global Term in the Boundary Conditions for Laplace Equation

Yelena Mustafayeva, Nihan Aliyev

Baku State University, Baku Azerbaijan
(e-mail: yelenamustafayeva@bsu.edu.az)

Problem Statement. We have considered the proof of the Fredholm property of some nonlocal boundary problems in [1-3]. We consider the three-dimensional Laplace equation in a bounded domain with a Lyapunov boundary:

$$\Delta u(x) = 0, \quad x \in D, \quad (1)$$

with a spectral parameter in nonlocal boundary conditions containing a global term:

$$\begin{aligned} \sum_{j=1}^3 \left[\alpha_{ij}^{(1)}(x') \frac{\partial u(x)}{\partial x_j} \Big|_{x_3=\gamma_1(x')} + \alpha_{ij}^{(2)}(x') \frac{\partial u(x)}{\partial x_j} \Big|_{x_3=\gamma_2(x')} \right] = \\ = \lambda \sum_{k=1}^2 \int_S \alpha_k(\xi') u(\xi', \gamma_k(\xi')) d\xi', \quad x' \in S, \end{aligned} \quad (2)$$

where $S = proj_{Ox_1x_2} D$, Γ_1 and Γ_2 are upper and lower half-surfaces of the boundary Γ respectively: $\Gamma_k = \{\xi = (\xi_1, \xi_2, \xi_3) : \xi_3 = \gamma_k(\xi'), \xi' = (\xi_1, \xi_2) \in S\}$, $\xi_3 = \gamma_k(\xi_1, \xi_2)$, $k = 1, 2$, are the equations of the halfsurfaces Γ_1 and Γ_2 ; functions $\gamma_k(\xi')$, $k = 1, 2$, are twice continuously differentiable with respect to both variables ξ_1, ξ_2 ; the coefficients $\alpha_{ij}^{(k)}(x')$, $\alpha_k(x')$, $i, k = 1, 2$; $j = \overline{1, 3}$, are continuous functions.

The fundamental solution of equation (1) is [4]

$$U(x - \xi) = -\frac{1}{4\pi} \frac{1}{|x - \xi|}. \quad (3)$$

Basic relationships and necessary conditions

Multiplying equation (1) by the fundamental solution (3), integrating it over the domain D and taking into account that $\Delta_x U(x - \xi) = \delta(x - \xi)$ where $\delta(x - \xi)$ is the Dirac δ -function we'll get **the first basic relationship**:

$$\begin{aligned} -\sum_{j=1}^3 \int_{\Gamma} \left[\left(\frac{\partial u(x)}{\partial x_j} U(x - \xi) - u(x) \frac{\partial U(x - \xi)}{\partial x_j} \right) \cos(\nu, x_j) dx \right] = \\ = \int_D u(x) \delta(x - \xi) dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2} u(\xi), & \xi \in \Gamma. \end{cases} \end{aligned} \quad (4)$$

Here the first relationship gives the representation of the general solution of equation (1) and the second expression in (4) is the **first necessary condition**.

Theorem 1. *Let a convex along the direction x_3 domain $D \subset R^3$ be bounded with the boundary Γ which is a Lyapunov surface. Then the obtained first necessary condition in (4) is regular.*

Multiplying (1) by $\frac{\partial U(x - \xi)}{\partial x_i}$, $i = \overline{1, 3}$, integrating it over the domain D we obtain the rest of **three basic relationships**:

$$\begin{aligned} \int_{\Gamma} \frac{\partial u(x)}{\partial x_i} \frac{\partial U(x - \xi)}{\partial \nu_x} dx + \\ + \int_{\Gamma} \frac{\partial u(x)}{\partial x_m} \left[\frac{\partial U(x - \xi)}{\partial x_i} \cos(\nu_x, x_m) - \frac{\partial U(x - \xi)}{\partial x_m} \cos(\nu_x, x_i) \right] dx + \\ + \int_{\Gamma} \frac{\partial u(x)}{\partial x_l} \left[\frac{\partial U(x - \xi)}{\partial x_i} \cos(\nu_x, x_l) - \frac{\partial U(x - \xi)}{\partial x_l} \cos(\nu_x, x_i) \right] dx = \\ = \begin{cases} -\frac{\partial u(\xi)}{\partial \xi_i}, & \xi \in D, \\ -\frac{1}{2} \frac{\partial u(\xi)}{\partial \xi_i}, & \xi \in \Gamma, \end{cases} \end{aligned} \quad (5)$$

$i = \overline{1, 3}$, the numbers i, m, l make a permutation of numbers 1, 2, 3.

The second expressions in (5) are the other three **necessary conditions** ($\xi \in \Gamma, i = \overline{1, 3}$) which taking into account that $\frac{\partial U(x-\xi)}{\partial x_i} = -\frac{x_i - \xi_i}{4\pi|x-\xi|^3} = -\frac{\cos(x-\xi, x_i)}{4\pi|x-\xi|^2}$ and introducing the designations

$$K_{ij}^{(k)}(x', \xi') = (\cos(x - \xi, x_i) \cos(\nu_x, x_j) - \cos(x - \xi, x_j) \cos(\nu_x, x_i)) \Bigg|_{\substack{x_3 = \gamma_k(x') \\ \xi_3 = \gamma_k(\xi')}} =$$

$$P_k(x', \xi') = 1 + \sum_{m=1}^2 \left(\frac{\partial \gamma_k(x')}{\partial x_m} \right)^2 \cos^2(x' - \xi', x_m) + O(|x' - \xi'|)$$

can be rewritten in the form

$$\begin{aligned} & \frac{1}{2} \frac{\partial u}{\partial \xi_i} \Big|_{\xi_3 = \gamma_k(\xi')} = \\ & = (-1)^{(k)} \int_S \frac{\partial u(x)}{\partial x_m} \Big|_{x_3 = \gamma_k(x')} \frac{1}{4\pi|x' - \xi'|^2} \frac{K_{im}^{(k)}(x', \xi')}{P_k(x', \xi')} \frac{dx'}{\cos(\nu_x, x_3)} + \\ & + (-1)^{(k+1)} \int_S \frac{\partial u(x)}{\partial x_l} \Big|_{x_3 = \gamma_k(x')} \frac{1}{4\pi|x' - \xi'|'^2} \frac{K_{il}^{(k)}(x', \xi')}{P_k(x', \xi')} \frac{dx'}{\cos(\nu_x, x_3)} + \dots, \\ & i = 1, 2, 3; k = 1, 2. \end{aligned} \quad (6)$$

Theorem 2. *Under assumptions of Theorem 1 necessary conditions (6) are singular.*

References

- [1] Y.Y Mustafayeva., N.A. Aliyev, New Method of Solvability of a Three-dimensional Laplace Equation with Nonlocal Boundary Conditions. Journal of Mathematical Physics, Analysis, Geometry, Kharkov, 2016, vol. 12, No. 3, pp. 185-204.

- [2] Y.Y.Mustafayeva, N.A. Aliyev, New method of investigation of the Fredholm property of three-dimensional Helmholtz equation with nonlocal boundary value conditions, J. Filomat, 2018, vol.32, issue 12, pp. 4445-4460.
- [3] Y.Y.Mustafayeva, Fredholm property of nonlocal BVP for integro-differential equation of elliptic type in cube, Azerbaijan Journal of Mathematics, V. 5, No 2, July, 2025, p.37-50.
- [4] V.S. Vladimirov, *Equations of Mathematical Physics*, Moscow: Mir, 1981.

On a Necessary Optimality Condition for a Linear Nonlocal Discrete Control Problem with a Nondifferentiable Objective Functional

Malahat Najafova

Institute of Mathematics, Baku, Azerbaijan
(e-mail: nacamelahet01@gmail.com)

Abstract: This paper considers a linear nonlocal discrete optimal control problem with a nondifferentiable performance criterion and establishes a necessary optimality condition in terms of the directional derivative.

Keywords: necessary optimality condition, nonlocal discrete optimal control problem, admissible control, directional derivative.

Let $T = \{t_0, t_0 + 1, \dots, t_1 - 1\}$ be a given discrete interval, $U \subset R^r$ a given nonempty and bounded set, $u(t)$ an r -dimensional discrete vector of control inputs, and $\varphi(a, b)$ a given scalar function satisfying the Lipschitz condition and having derivatives in the directions [1].

It is clear that the class of functions that have derivatives in all directions is broader than the class of continuously differentiable functions.

Consider the problem of minimizing the functional

$$J(u) = \varphi(x(t_0), x(t_1)), \quad (1)$$

subject to the constraints

$$u(t) \in U \subset R^r, t \in T, \quad (2)$$

$$x(t+1) = A(t)x(t) + f(t, u(t)), t \in T, \quad (3)$$

$$L_0 x(t_0) + L_1 x(t_1) = \ell. \quad (4)$$

Here, $A(t)$ is a given $(n \times n)$ discrete matrix function, L_0 and L_1 are given $(n \times n)$ constant matrices, ℓ is a given constant vector, and $f(t, u)$ is a given n -dimensional vector function that is continuous over the set of variables.

We will refer to any control function with the properties listed above as a feasible control, and the corresponding process $(u(t), x(t))$ as a feasible process.

We will refer to the admissible process $(u(t), x(t))$, which is a solution to the given problem (problems (1)-(4)), as the optimal process.

Let $(u(t), x(t))$ be a valid process.

Let $(\hat{u}(t) = u(t) + \Delta u(t), \hat{x}(t) = x(t) + \Delta x(t))$ denote an arbitrary valid process, and let us write the increment of the quality functional as

$$\Delta J(u) = J(\hat{u}) - J(u) = \varphi(\hat{x}(t_0), \hat{x}(t_1)) - \varphi(x(t_0), x(t_1)). \quad (5)$$

Let us introduce the notation

$$\Delta_{\hat{u}(t)} f[t] \equiv f(t, \hat{u}(t)) - f(t, u(t)).$$

Suppose that the set (the set of admissible velocities)

$$f(t, U) = \{\alpha : \alpha = f(t, v(t)), v(t) \in U, t \in T\}. \quad (6)$$

is convex, $\varepsilon \in [0, 1]$ is an arbitrary number, and $v(t), t \in T$ is an arbitrary admissible control. Then the special increment of the control $u(t)$ can be defined by the formula [1]

$$\Delta u(t; \varepsilon) = v(t; \varepsilon) - u(t),$$

where $v(t; \varepsilon), t \in T$ is an arbitrary vector such that

$$\Delta_{v(t; \varepsilon)} f[t] = \varepsilon \Delta_{v(t)} f[t]. \quad (7)$$

After some transformations, taking into account the formula for the directional derivative, we obtain the validity of the expansion

$$\begin{aligned} \Delta J_\varepsilon(u) &= \varphi(x(t_0) + \varepsilon q_1(v), x(t_1) + \varepsilon q_2(v)) - \varphi(x(t_0), x(t_1)) = \\ &= \varepsilon \left[\frac{\partial \varphi(x(t_0), x(t_1))}{\partial q_1(v)} + \frac{\partial \varphi(x(t_0), x(t_1))}{\partial q_2(v)} \right] + o(\varepsilon). \end{aligned} \quad (8)$$

Here,

$$q_1(v) = [L_0 + L_1 F(t_1, t_0 - 1)]^{-1} L_1 \sum_{\tau=t_0}^{t_1-1} F(t_1, \tau) \Delta_{v(\tau)} f[\tau],$$

$$q_2(v) = [-F(t_1, t_0 - 1) [L_0 + L_1 F(t_1, t_0 - 1)]^{-1} L_1 + E] \sum_{\tau=t_0}^{t_1-1} F(t_1, \tau) \Delta_{v(\tau)} f[\tau],$$

$F(t, \tau)$ is an $(n \times n)$ matrix function (analogous to the Cauchy matrix [2]) that is a solution to the problem

$$F(t, \tau - 1) = F(t, \tau) A(t),$$

$$F(t, t - 1) = E,$$

($E - (n \times n)$ identity matrix).

It follows from the expansion (8) that the statement holds

Theorem 1. If set (6) is convex, then for the feasible control $u(t)$ to be optimal, it is necessary that the inequality

$$\frac{\partial \varphi(x(t_0), x(t_1))}{\partial q_1(v)} + \frac{\partial \varphi(x(t_0), x(t_1))}{\partial q_2(v)} \geq 0 \quad (9)$$

was performed for all $v(t) \in U, t \in T$.

References

- [1] K.B. Mansimov, *Discrete Systems*. Baku: Baku State University Press, 2013, 151 pp.
- [2] K.B. Mansimov, M.Y. Najafova On a Problem of Optimal Control of Discrete Systems with Nonlinear Boundary Conditions. *Proceedings of the International Conference Dedicated to the 55th Anniversary of the Institute of Mathematics and Mechanics*. Baku, May 15–16, 2014, pp. 285–286

On the Triangular Representation of the Jost-Type Solution to the Perturbed Modified Mathieu Equation

Gunay Rzayeva

Sumgayit State University, Sumgayit, Azerbaijan

(e-mail: gunay.rzayeva@sdu.edu.az)

The Mathieu differential equation often arises in solving scientific and engineering problems (see [1], [3], [6]). The most interesting example is the problem of oscillations of an elliptical membrane (see [3]). The Mathieu equations also arise when studying the propagation of electromagnetic waves in an elliptical cylinder, when considering surface waves of a liquid in a vessel having the shape of an elliptical cylinder, and when solving a number of other issues (see [6]).

Consider the following perturbed modified Mathieu equation

$$-y'' + (chx)y + q(x)y = \lambda y, 0 < x < \infty, \lambda \in C, \quad (1)$$

where the real potential $q(x)$ satisfies the conditions

$$q(x) \in C^{(1)}[0, \infty], \quad \int_0^\infty e^x |q(x)| dx < \infty. \quad (2)$$

Note that equation (1) is a one-dimensional Schrödinger equation with an additional exponentially growing potential. The last equation is of particular interest from a physical point of view.

It is known [6] that equation (1) with $q(x) = 0$ has a special solution $f_0(x, \lambda)$, which can be represented as

$$f_0(x, \lambda) = \left(\sqrt{2}e^{\frac{x}{2}}\right)^{-1} e^{-\sqrt{2}e^{\frac{x}{2}}} \sum_{r=0}^{\infty} (-1)^r c_r \left(\sqrt{2}e^{\frac{x}{2}}\right)^{-r}, \quad (3)$$

where

$$\begin{aligned} c_0 &= 1, & c_1 &= \frac{4\lambda + 1^2}{8}, & c_2 &= \frac{(4\lambda + 1^2)(4\lambda + 3^2)}{8^2 2!}, \\ c_3 &= \frac{(4\lambda + 1^2)(4\lambda + 3^2)(4\lambda + 5^2)}{8^3 3!} - \frac{4}{3!}, \end{aligned} \quad (4)$$

$$c_4 = \frac{(4\lambda + 1^2)(4\lambda + 3^2)(4\lambda + 5^2)(4\lambda + 7^2)}{8^4 4!} - \frac{2}{4!}(4\lambda + 13), \dots$$

Moreover, for each x the solution $f_0(x, \lambda)$ is an entire function with respect to λ . Note that, by virtue of (3), (4), for each fixed λ the function $f_0(x, \lambda)$ belongs to the space $L_2(0, +\infty)$. Of particular interest is the Jost-type solution of the perturbed equation (1), i.e., solution $f(x, \lambda)$ that satisfies the condition $f(x, \lambda) = f_0(x, \lambda)[1 + o(1)]$, $x \rightarrow \infty$.

In the present paper, by means of the transformation operator, we find a representation of a special Jost-type solution $f(x, \lambda)$ of equation (1). When obtaining an integral equation for the kernel of the representation, it turned out to be natural to use the Riemann function method. The results of this paper can be used to study various spectral problems for equation (1).

Note that a transformation operator with a condition at infinity is constructed in [2], [7] for the one-dimensional Schrödinger equation with a rapidly decreasing potential. At the same time, for the Schrödinger equations with unbounded potentials, the construction of transformation operators encounters significant difficulties in comparison with a rapidly decreasing potential. In this direction, we note the works [2], [7], in which were studied the transformation operators for the Schrödinger equation with an additional potential of the form cx^α , $\alpha = 1, 2$.

Note that the usual Mathieu equations are special cases of the Hill equation. For perturbations of the last equation, the transformation operators are studied in detail (see [9] and references therein)

Let

$$s_0(x) = \int_x^\infty |q(t)| dt, \quad s_1(x) = \int_x^\infty s_0(t) dt. \quad (5)$$

The main result of the present paper is as follows

Theorem 1. *If the potential $q(x)$ satisfies condition (1), then for all values of λ , the equation (1) has solution $f(x, \lambda)$ representable as*

$$f(x, \lambda) = f_0(x, \lambda) + \int_x^{+\infty} K(x, t) f_0(t, \lambda) dt \quad (6)$$

where the kernel $K(x, t)$ is a continuously differentiable function and satisfies the following conditions:

$$|K(x, t)| \leq \frac{1}{2} s_0\left(\frac{x+t}{2}\right) e^{s_1(x)},$$

$$K(x, x) = \frac{1}{2} \int_x^{+\infty} q(t) dt.$$

References

- [1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Natl. Bur. Stand. Appl. Math. Ser. 55, U.S. Government Printing Office, Washington, 1964.
- [2] M.G. Gasymov and B.A. Mustafaev, Inverse scattering problem for an anharmonic equation on the semiaxis. *Dokl. Akad. Nauk SSSR*, **228** (1976), no.1, 11–14.
- [3] C. Hunter and B. Guerrieri, The eigenvalues of ME and their branch points, *Studies in Appl. Math.*, **64** (1981), 113–141.
- [4] A.V. Kashevarov, Mathieu functions and coulomb spheroidal functions in the electrostatic probe theory, *Comput. Math. Math. Physics*, **51** (2011), 2137–2145.
- [5] V. A. Marchenko, *Sturm–Liouville Operators and Applications. Translated from Russian by A. Jacob*, Oper. Theory Adv. Appl. 22, Birkhauser, Basel, 1986.
- [6] N.W. McLachlan, *Theory and application of Mathieu functions*, Oxford., 1947.
- [7] D.H. Orucov, On the transformation operator for the Schrodinger equation with an additional increasing potential, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **48** (2022), no.2, 311–322.

On Asymptotics of Solution of Boundary Value Problem for Nonclassical Type Singularly Perturbed Quasilinear Differential Equation of Arbitrary Odd Order

Mahir Sabzaliev, Ilhama Sabzalieva

Azerbaijan State Oil and Industry University, Baku, Azerbaijan
(e-mail: sabzaliev@mail.ru, isabzalieva@mail.ru)

The study of numerous real phenomena characterized by nonuniform transitions between different physical states requires the investigation of boundary-value problems for differential equations with a small parameter multiplying the highest order derivatives. Such problems, known as singularly perturbed problems, have attracted the attention of many mathematicians ([1], [5], [6]). However, most of the studied singularly perturbed partial differential equations belong to one of the three classical types. The investigation of boundary value problems for nonclassical type singularly perturbed differential equations requires a specific approach to their solution. Let's note here some works devoted to the investigation of boundary –value problems for nonclassical type singularly perturbed differential equations [2],[3],[4].

In the paper the following boundary value problem is considered in the rectangle $D = \{(t, x) | 0 \leq t \leq 1, 0 \leq x \leq 1\}$:

$$(-1)^m \varepsilon^{2m} \frac{\partial^{2m+1} u}{\partial t^{2m+1}} - \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + F(t, x, u) = 0, \quad (1)$$

$$u|_{t=0} = \frac{\partial u}{\partial t} \Big|_{t=0} = \dots = \frac{\partial^m u}{\partial t^m} \Big|_{t=0} = 0, \quad (0 \leq x \leq 1) \quad (2)$$

$$\frac{\partial^{m+1} u}{\partial t^{m+1}} \Big|_{t=1} = \frac{\partial^{m+2} u}{\partial t^{m+2}} \Big|_{t=1} = \dots = \frac{\partial^{2m} u}{\partial t^{2m}} \Big|_{t=1} = 0, \quad (0 \leq x \leq 1) \quad (3)$$

$$u|_{x=0} = u|_{x=1} = 0, \quad (0 \leq t \leq 1). \quad (4)$$

Here $\varepsilon > 0$ is a small parameter, m is arbitrary natural number, $F(t, x, u)$ is a given function that may depend on u either linearly i.e. $F(t, x, u) = a(t, x)u - f(t, x)$, or nonlinearly.

Our goal is to construct a complete asymptotics of the solution of boundary value problem (1)-(4) with respect to the small parameter. For this purpose,

iterative processes were carried out, and as a result, the following theorem was proved.

Theorem. Let the function $F(t, x, u) \in C^{2m+2n+2}(D \times (-\infty, +\infty))$, satisfy the conditions $\frac{\partial F(t, x, u)}{\partial u} \geq \gamma^2 > 0$ at $(t, x, u) \in (D \setminus \{(t, x) \in D | t = x\}) \times (-\infty, +\infty)$, and also the condition

$$\left. \frac{\partial^i f(t, x)}{\partial t^{i_1} \partial x^{i_2}} \right|_{t=x} = 0; \quad i = i_1 + i_2; i = 0, 1, \dots, 2n + 2$$

in the case of linear dependence F on u and the condition

$$F(t, x, u)|_{t=x} = 0, \quad \left. \frac{\partial^i F(t, x, 0)}{\partial t^{i_1} \partial x^{i_2} \partial u^{i_3}} \right|_{t=x} = 0; \quad i = i_1 + i_2 + i_3; \quad i = 0, 1, \dots, 2n + 2$$

in the case of nonlinear dependence F on u . Then for the solution of the boundary value problem (1)-(4) the following asymptotic representation holds

$$u = \sum_{i=0}^n \varepsilon^i W_i + \sum_{s=0}^{n+m-1} \varepsilon^{1+s} V_s + \sum_{s=0}^{n+m-1} \varepsilon^{1+m+s} \eta_s + \sum_{j=0}^{n+1} \varepsilon^j \psi_j + \varepsilon^{n+1} z.$$

Here the functions W_i are defined by the first iterative process, V_s, η_s, ψ_j are boundary layer type functions close to boundaries $t = 0, t = 1, x = 1$, which are defined by the corresponding iterative processes, and $\varepsilon^{n+1}z$ is remainder term, moreover, for the function z the following estimation holds:

$$\varepsilon^m \left\| \frac{\partial^m z}{\partial t^m} \right\|_{t=1} \Big\|_{L_2(0,1)} + \varepsilon^{\frac{1}{2}} \left\| \frac{\partial z}{\partial x} \right\|_{L_2(D)} + c_1 \|z\|_{L_2(D)} \leq c_2,$$

where $c_1 > 0, c_2 > 0$ are constants, not depending on ε .

References

- [1] Lomov, S.A.: Introduction to general theory of singular perturbations, M.: Nauka, 1981 398 p.
- [2] Y.Sh.Salimov, I.M.Sabzaliyeva, On boundary value problems for one class of nonclassical type singular perturbed differential equations, Doklady Akad.Nauk. Rossii, 399(4), 2004, 450-453.

- [3] Y.Sh.Salimov, I.M.Sabzaliyeva, On boundary value problems for one class of arbitrary odd order singular perturbed equations, *Differential equations* 42(5), 2006, 653-659.
- [4] M.M. Sabzaliev, I.M. Sabzalieva, asymptotics of solutions of a boundary value problem for quasilinear non-classical type differential equation of arbitrary odd order, *British Journal of Mathematics. Computer Science*, 22(4) 2017, 1-19. Article no BJMCS 33704 ISSN: 2231-0851
- [5] Vishik, M.I., Lyusternik, L.A.: Regular degeneration and boundary layer for linear differential equations with a small parameter, *Uspekhi Math. Nauk.* 12 5(77), 3-122 (1957)
- [6] Vazov, V.: Asymptotic expansion of solutions to ordinary differential equation, M.: Mir, (1968).

A Duality Framework for Optimal Control of Hyperbolic Differential Inclusions

Sevilay Demir Sağlam^a, Elimhan Mahmudov^{b,c}

^a*Istanbul University, Department of Mathematics, Istanbul, Türkiye*

^b*Azerbaijan National Aviation Academy, Baku, Azerbaijan*

^c*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
(e-mail: sevilay.demir@istanbul.edu.tr, elimhan22@yahoo.com)

The optimal control of hyperbolic differential inclusions is vital for modeling non-smooth wave and vibration dynamics. While foundational variational tools were established by Aubin and Cellina [1] and Mordukhovich [4], the locally adjoint mapping (LAM) formalism [3] provided the essential link between discrete approximations and continuous-time systems. Building on previous duality studies for various boundary value problems, this study extends the LAM paradigm to a class of constrained hyperbolic inclusions. We derive rigorous sufficient optimality conditions and construct a comprehensive conjugate duality framework tailored to the wave operator's unique transversality mechanisms. By verifying a zero duality gap through an explicitly solvable one-dimensional wave equation, we establish this framework as a powerful and

robust alternative to conventional primal methodologies in non-smooth hyperbolic control.

We consider the primal optimal control problem (P) governed by a hyperbolic differential inclusion:

$$\begin{aligned} \min_u \quad J(u) &= \iint_D f(u(x, t), x, t) dx dt + \int_0^1 \varphi(u(x, 1)) dx \\ u_{tt}(x, t) - u_{xx}(x, t) &\in F(u(x, t), x, t), \quad \text{a.e. } (x, t) \in D = [0, 1]^2, \quad (P) \\ u(x, t) &\in \phi(x, t), \quad \forall (x, t) \in D, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0, \quad u(0, t) = 0, \quad u(1, t) = 0. \end{aligned}$$

We assume homogeneous initial and boundary conditions for operator invertibility. The state $u \in W_1^{2,2}(D, \mathbb{R}^n)$ satisfies $u_{tt} - u_{xx} \in F(u, x, t)$ and $u \in \phi(x, t)$, where F and ϕ are convex. The objective $J(u)$ consists of a Carathéodory running cost f and a terminal cost φ .

Defining the Hamiltonian $H_F(u, v^*) = \sup_{v \in F(u)} \langle v, v^* \rangle$, the corresponding argmaximum set $F_A(u; v^*) = \{v \in F(u) : \langle v, v^* \rangle = H_F(u, v^*)\}$, and the locally adjoint mapping $F^*(v^*; w_0) = \{u^* : (u^*, -v^*) \in K_{\text{gph } F}^*(w_0)\}$, we establish the conjugate linkage by the functional $M_F(u^*, v^*, x, t) = \inf_u \{\langle u^*, u \rangle - H_F(u, v^*, x, t)\}$.

Theorem 1. Let φ and f be proper, continuous, convex Carathéodory functions, and let F and ϕ be convex-valued and closed. For the optimality of a feasible state trajectory $\tilde{u}(x, t)$ of the continuous problem, it is sufficient that an adjoint function $u^*(x, t) \in W_1^{2,2}(D, \mathbb{R}^n)$ satisfies a.e. on D :

$$\begin{aligned} (i) \quad & u_{tt}^*(x, t) - u_{xx}^*(x, t) \in F^*(u^*(x, t); (\tilde{u}(x, t), \tilde{u}_{tt}(x, t) - \tilde{u}_{xx}(x, t))) \\ & \quad - \partial_u f(\tilde{u}(x, t), x, t), \\ (ii) \quad & u_t^*(x, 1) \in \partial \varphi(\tilde{u}(x, 1)) - K_{\phi(x, 1)}^*(\tilde{u}(x, 1)), \quad \forall x \in (0, 1), \\ & u^*(x, 1) = 0, \quad \forall x \in (0, 1), \quad u^*(0, t) = u^*(1, t) = 0, \quad \forall t \in (0, 1), \\ (iii) \quad & \tilde{u}_{tt}(x, t) - \tilde{u}_{xx}(x, t) \in F_A(\tilde{u}(x, t); u^*(x, t)). \end{aligned}$$

We formulate the exact continuous dual problem (D) using convex duality:

$$\sup_{\substack{u^*, p^*, q^* \\ u^*(0, t) = u^*(1, t) = 0, \quad u^*(x, 1) = 0}} J_*\left(u^*(x, t), p^*(x, t), q^*(x, t)\right) \quad (D)$$

where $J_* = \iint_D \left[M_F(u_{tt}^*(x, t) - u_{xx}^*(x, t) - p^*(x, t) + q^*(x, t), u^*(x, t), x, t) - W_{\phi(x, t)}(-p^*(x, t)) - f^*(q^*(x, t), x, t) \right] dxdt - \int_0^1 \varphi^*(u_t^*(x, 1)) dx.$

Here, M_F absorbs the inclusion dynamics, f^* and φ^* are Fenchel conjugates, W_ϕ is the support function, and the variables p^* and q^* dualize the state constraints and running cost, respectively.

Theorem 2. (Weak Duality) For all feasible primal solutions $u(x, t)$ and dual solutions (u^*, p^*, q^*) , the inequality $J(u) \geq J_*(u^*, p^*, q^*)$ holds.

Theorem 3. (Strong Duality) Suppose $\tilde{u}(x, t)$ satisfies Theorem 1, and the dual variables (u^*, p^*, q^*) satisfy the transversality conditions: $W_\phi(-p^*) = \langle \tilde{u}, -p^* \rangle$, $q^* \in \partial_u f(\tilde{u})$, and $\varphi^*(u_t^*(x, 1)) = \langle u_t^*(x, 1), \tilde{u}(x, 1) \rangle - \varphi(\tilde{u}(x, 1))$. Then $\tilde{u}$ and (u^*, p^*, q^*) are exact optimal solutions for (P) and (D), respectively, achieving a zero duality gap ($J = J_*$).

Example 1. For the primal problem with $F = \phi = [-1, 1]$ and objective $J(u) = \int_0^1 -u(x, 1) \sin(\pi x) dx$, the optimality conditions yield the adjoint state $u^*(x, t) = \frac{1}{\pi} \sin(\pi t) \sin(\pi x)$. Since $u^* > 0$ internally, maximizing the Hamiltonian gives $\tilde{u}_{tt} - \tilde{u}_{xx} = 1$, resulting in the optimal primal trajectory:

$$\tilde{u}(x, t) = \sum_{k=1,3,5,\dots}^{\infty} \frac{4}{(k\pi)^3} (1 - \cos(k\pi t)) \sin(k\pi x).$$

With a maximum amplitude ≈ 0.27 , the state constraint remains inactive ($p^* = 0$). Consequently, the exact dual problem simplifies to:

$$\sup_{u^*} J_*(u^*) = \iint_D -u^*(x, t) dxdt$$

subject to $u_{tt}^* - u_{xx}^* = 0$, $u^*(x, 1) = 0$, and $u_t^*(x, 1) = -\sin(\pi x)$. Evaluating both objective functionals, the identity $J(\tilde{u}) = J_*(u^*)$ strictly confirms the zero duality gap, validating Theorem 3.

$$J(\tilde{u}) = \int_0^1 - \left[\sum_{k \text{ odd}} \frac{8}{(k\pi)^3} \sin(k\pi x) \right] \sin(\pi x) dx = -\frac{4}{\pi^3},$$

$$J_*(u^*) = \iint_D -\frac{1}{\pi} \sin(\pi t) \sin(\pi x) dx dt = -\frac{4}{\pi^3}.$$

References

- [1] J.P.Aubin, A.Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.
- [2] S. Demir Sağlam, *Duality in the problems of optimal control described by Darboux-type differential inclusions*, Optim. Lett., 2024, 18, 1811-1835.
- [3] E.N. Mahmudov, *Approximation and Optimization of Discrete and Differential Inclusions*, Elsevier, Boston, 2011.
- [4] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications*, Springer, Berlin, 2006.

Optimal Boundary Control for Plates in the Special Cases

Khayala Seyfullayeva

Sumgait State University, Sumgait, Azerbaijan
(e-mail: xeyaleseyfullayeva82@gmail.com)

Let the controlling process be described by the equation of the thin plate vibrations

$$\frac{\partial^2 u}{\partial t^2} + a^2 \Delta^2 u = 0 \text{ in } Q_T = \Omega \times (0, T), \Omega = (0, l_1) \times (0, l_2) \quad (1)$$

with initial

$$u(x_1, x_2, 0) = \varphi_0(x_1, x_2), \frac{\partial u(x_1, x_2, 0)}{\partial t} = \varphi_1(x_1, x_2), (x_1, x_2) \in \Omega \quad (2)$$

and boundary conditions

$$u(0, x_2, t) = v(x_2, t), u(l_1, x_2, t) = 0, \frac{\partial u(0, x_2, t)}{\partial x_1} = 0, \frac{\partial u(l_1, x_2, t)}{\partial x_1} = 0,$$

$$(x_2, t) \in (0, l_2) \times (0, T), \quad u(x_1, 0, t) = 0, \quad u(x_1, l_2, t) = 0, \quad \frac{\partial u(x_1, 0, t)}{\partial x_2} = 0, \\ \frac{\partial u(x_1, l_2, t)}{\partial x_2} = 0, \quad (x_1, t) \in (0, l_1) \times (0, T), \quad (3)$$

where a^2, l_1, l_2, T are given positive numbers, $v(x_2, t)$ is a boundary control function, $\varphi_0(x_1, x_2) \in W_2^2(\Omega)$, $\varphi_1(x_1, x_2) \in L_2(\Omega)$ are given functions, Δ is Laplace operator with respect to x_1, x_2 .

Consider the space of controls $H = W_2^{4,2}((0, l_2) \times (0, T))$.

As a class of admissible controls U_{ad} we consider the set of functions $v(x_2, t)$ from H , for which hold: $v(0, t) = v(l_2, t) = 0$, $\frac{\partial v(0, t)}{\partial x_2} = \frac{\partial v(l_2, t)}{\partial x_2} = 0$, $v(x_2, 0) = \frac{\partial v(x_2, 0)}{\partial t} = 0$, moreover $\left\| \frac{\partial^4 v}{\partial x_2^4} \right\|_{L_2((0, l_2) \times (0, T))} \leq M_1$, $\left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L_2((0, l_2) \times (0, T))} \leq M_1$, where M_1 is a given number.

Here and further on by M_i we denote different constants not depending on the admissible controls and estimating quantities.

It is supposed that the functions $\varphi_0(x_1, x_2)$ and $v(x_2, t)$ satisfy the proper conditions.

The problem is: On the set U_{ad} to find a function that gives minimum to the functional

$$J(v) = \frac{1}{2} \int_{\Omega} [u(x_1, x_2, T)]^2 dx_1 dx_2 + \frac{\alpha}{2} \int_0^{l_2} \int_0^T v^2(x_2, t) dx_2 dt, \quad (4)$$

Subject to (1)-(3), where $\alpha > 0$ is a positive number.

As a solution of the problem (1)-(3) for each fixed admissible control $v(x_2, t)$ we take the function $u(x_1, x_2, t) \in W_2^{2,1}(Q_T)$, which for any function $\eta \in W_2^{2,1}(Q_T)$, $\eta(x_1, x_2, T) = 0$,

$$\eta(0, x_2, t) = 0, \quad \eta(l_1, x_2, t) = 0, \quad \frac{\partial \eta(0, x_2, t)}{\partial x_1} = 0, \quad \frac{\partial \eta(l_1, x_2, t)}{\partial x_1} = 0, \\ \eta(x_1, 0, t) = 0, \quad \eta(x_1, l_2, t) = 0, \quad \frac{\partial \eta(x_1, 0, t)}{\partial x_2} = 0, \quad \frac{\partial \eta(x_1, l_2, t)}{\partial x_2} = 0$$

Satisfies to the integral identity

$$\int_{Q_T} \left[-\frac{\partial u}{\partial t} \cdot \frac{\partial \eta}{\partial t} + a^2 \Delta u \Delta \eta \right] dx_1 dx_2 dt - \int_{\Omega} \varphi_1(x_1, x_2) \eta(x_1, x_2, 0) dx_1 dx_2 = 0$$

and conditions

$$u(x_1, x_2, 0) = \varphi_0(x_1, x_2), u(0, x_2, t) = v(x_2, t)$$

in the usual sense.

Note that the theorem may be proved on the existence and uniqueness of the solution $u(x_1, x_2, t)$ of the boundary problem (1)-(3) by each fixed admissible control $v(x_2, t)$ and this solution has a property $u \in C([0, T]; W_2^2(\Omega))$, $u \in C([0, T]; L_2(\Omega))$.

The following results were obtained:

Theorem 1. Let the conditions above on the data of the problem (1)-(4) be satisfied. Then there exists the unique optimal control in the problem (1)-(4).

Let us introduce the adjoint problem for the control

$$\frac{\partial^2 \psi}{\partial t^2} + a^2 \Delta^2 \psi = 0 \text{ in } Q_T, \quad (5)$$

$$\psi(x_1, x_2, T) = 0, \frac{\partial \psi(x_1, x_2, T)}{\partial t} = -u(x_1, x_2, T), (x_1, x_2) \in \Omega, \quad (6)$$

$$\psi(0, x_2, t) = \psi(l_1, x_2, t) = 0, \frac{\partial \psi(0, x_2, t)}{\partial x_1} = \frac{\partial \psi(l_1, x_2, t)}{\partial x_1} = 0,$$

$$(x_2, t) \in [0, l_2] \times [0, T],$$

$$\psi(x_1, 0, t) = \psi(x_1, l_2, t) = 0, \frac{\partial \psi(x_1, 0, t)}{\partial x_2} = 0, \frac{\partial \psi(x_1, l_2, t)}{\partial x_2} = 0,$$

$$(x_1, t) \in [0, l_1] \times [0, T], \quad (7)$$

where $u(x_1, x_2, t)$ is a solution of the problem (1)-(3) for the given control $v(x_2, t)$.

Since $u(x_1, x_2, T) \in W_2^2(\Omega)$, the adjoint problem (5)-(7) has the only solution in the space $W_2^{4,2}(Q_T)$.

Theorem 2. Let the conditions above on the data of the problem (1)-(4) be satisfied. Then for the optimality of the control $v_*(x_2, t) \in U_{ad}$ in the problem (1)-(4) it is necessary and sufficient fulfillment of the inequality

$$\int_0^T \int_0^{l_2} \left[\alpha v(x_2, t) - a^2 \left(\frac{\partial^3 \psi_*(0, x_2, t)}{\partial x_1^3} + \frac{\partial^3 \psi_*(0, x_2, t)}{\partial x_1 \partial x_2^2} \right) \right] \times$$

$$(v(x_2, t) - v_*(x_2, t)) dx_2 dt \geq 0 \quad \forall v \in U_{ad}$$

where $\psi_*(x_1, x_2, t)$ is a solution of the problem (5)-(7) by $u = u_*(x_1, x_2, t)$, and $u_*(x_1, x_2, t)$ is a solution of the problem (1)-(3) by $v = v_*(x_2, t)$.

In the proposed work, the problem of boundary optimal control for a linear equation of oscillations of a thin plate is considered. In the work, the theorem of existence and uniqueness of optimal control is proved, the differential of the functional is calculated and a necessary and sufficient condition for optimality in the form of an integral inequality is derived.

References

- [1] Arman J.-L.H. Application of the Optimal Control Theory with the Distributed Parameters Systems to the Optimal Design Problems, Moscow, Mir, 1977, 144 p.
- [2] Nasibzade V.N. On determination of Edgefunctions for equation of oscillations of rectangular membrane. Journal of Automation and Information Sciences, 49(5):, 2017, pp.72-81.

Nonlocal Navier-Stokes Equations and Applications

Veli Shakhmurov

Antalya Bilim University, Antalya, Türkiye
(e-mail: veli.sahmurov@antalya.edu.tr)

In this talk, the Cauchy problem for the stationary and instationary nonlocal incompressible abstract Navier-Stokes equation is considered. The equation involve a convolution term and an abstract operator function $A = A(x)$ in a Fourier type Banach space E . First, the existence, uniqueness and coercive estimates of corresponding Stokes equation is derived in L^p spaces. Then the existence and uniqueness of Navier-Stokes equation is obtained. We can obtain different classes of Stokes and Navier-Stokes equations by choosing the space E and the linear operator A which occur in a wide variety of physical systems. In application the existence, uniqueness and L^p -regularity properties

to solution of initial value problems for nonlocal degenerate Stokes equations and the existence and uniqueness of nonlocal Navier Stokes equations with discontinuous coefficients are established. We consider the Cauchy problem for the nonlocal Navier-Stokes(N-S) equation

$$\begin{aligned} \frac{\partial u}{\partial t} - a * \Delta u + Au + (u \cdot \nabla)u + \nabla \phi &= f(x, t), \\ \operatorname{div} u &= 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n; \quad t \in (0, T), \end{aligned} \quad (1)$$

where $A = A(x)$ is a linear operator function in a Banach space E .

$$u = (u_1(x, t), u_2(x, t), \dots, u_n(x, t)), \quad \varphi = \varphi(x, t), \quad u_0 = u_0(x)$$

are an E -valued vector solution, unknown pressure and data functions, respectively. Moreover,

$$f = (f_1(x, t), f_2(x, t), \dots, f_n(x, t))$$

is a given E -valued inhomogeneity vector function, $a_k = a_k(x)$ are complex-valued functions,

$$a = a(x) = (a_1(x), a_2(x), \dots, a_n(x)),$$

$$a \cdot u = (a_1 \cdot u_1, a_2 \cdot u_2, \dots, a_n \cdot u_n)$$

where $a_i \cdot u_i$ denotes the convolution of the functions a_i and u_i defined by

$$a_i \cdot u_i = \int_{\mathbb{R}^n} a_i(y) u_i(x - y, t) dy.$$

The symbol $(u \cdot \nabla)$ for $u = (u_1, u_2, \dots, u_n)$ is a vector function defined by

$$(u \cdot \nabla)u = \sum_{k=1}^n u_k \frac{\partial u}{\partial x_k} = \left(\sum_{k=1}^n u_k \frac{\partial u_1}{\partial x_k}, \sum_{k=1}^n u_k \frac{\partial u_2}{\partial x_k}, \dots, \sum_{k=1}^n u_k \frac{\partial u_n}{\partial x_k} \right).$$

Reduction of a Nonlocal Boundary Value Problem of a Second-Order Differential Equation to an Integral Equation

Yagub Sharifov^{a,b}, Nazaket Mammadova^a

^a*Baku State University, Baku Azerbaijan*

^b*Institute of Mathematics, Baku, Azerbaijan*

(e-mail: sharifov22@rambler.ru, nzkt.0304@mail.ru)

Mathematical modeling of several real phenomena leads to the occurrence of nonlinear boundary value problems of differential equations. In fact, the study of boundary value problems has developed into an important area of research over the years in view of their extensive applications in diverse disciplines such as fluid mechanics, mathematical physics, etc. For details, we refer the reader to the works [1-2]. In the analysis of nonlinear boundary value problems, we are primarily interested in examining the effect of nonlinearity on the solutions of the given problem. The available literature on the topic contains theoretical development of the topic as well as analytic and numerical methods for solving boundary value problems. Classical boundary conditions fail to take into account certain peculiarities of physical, chemical, or other processes happening inside the domain. This led to the birth of nonlocal conditions, which connect the boundary values of the unknown function to its values at some interior positions of the domain.

In this paper, we investigate the reduction of the solution of a second-order nonlinear differential equation with nonlocal boundary conditions to an integral equation.

Let us assume that for $t \in [0, T]$

$$x''(t) = f(t, x(t)), \quad (1)$$

$$\int_0^T m(t) x'(t) dt = 0, \quad (2)$$

$$x(0) = 0 \quad (3)$$

a system of differential equations is given. Here, $f: [0, T] \times R^n \rightarrow R^n$ is a given continuous function, and $m(t) \in R^{n \times n}$ are given matrices function.

Theorem. Let $f \in C([0, T] \times R^n, R^n)$ and $\det \int_0^T m(t) dt \neq 0$, the solution of the problem (1)-(3) is

$$x(t) = t \int_0^T K(t, t) f(t, x(t)) dt - \int_0^t t f(t, x(t)) dt, \quad (4)$$

where

$$K(t, t) = \overline{m}^{-1}(T) \begin{cases} \int_0^t m(t) dt, & 0 \leq t \leq t, \\ -\int_t^T m(t) dt, & t \leq t \leq T, \end{cases}$$

$$\overline{m}^{-1}(T) = \left(\int_0^T m(t) dt \right)^{-1}.$$

Proof. Let us integrate the equality (1) from 0 to t , we obtain

$$x'(t) = x'(0) + \int_0^t f(t, x(t)) dt$$

If we consider $x(t)$ in condition (2).

$$x'(0) \int_0^T m(t) dt + \int_0^T \int_t^T m(t) dt f(t, x(t)) dt = 0$$

using the equality, the $x'(0)$ vector can be defined. Then

$$x'(0) = -\overline{m}^{-1}(T) \int_0^T \int_t^T m(t) dt f(t, x(t)) dt.$$

We obtain

$$x'(t) = -\overline{m}^{-1}(T) \int_0^T \int_t^T m(t) dt f(t, x(t)) dt + \int_0^t f(t, x(t)) dt \quad (5)$$

From here

$$x'(t) = \int_0^T K(t, t) f(t, x(t)) dt.$$

Let's integrate equation (5) again from 0 to t

$$x(t) = \overline{m}^{-1}(T) \int_0^t \int_0^s \int_0^s m(a) da f(t, x(t)) ds dt -$$

$$\begin{aligned}
& -\overline{m}^{-1}(T) \int_0^t \int_s^T \int_t^T m(a) da f(t, x(t)) dt ds = \\
& = \overline{m}^{-1}(T) \int_0^t (t-s) \int_0^s m(a) da f(t, x(t)) ds - \\
& -\overline{m}^{-1}(T) \int_t^T s \int_s^T m(a) da f(s, x(s)) ds - t \int_t^T \int_s^T m(a) da f(s, x(s)) ds.
\end{aligned}$$

From here

$$\begin{aligned}
x(t) &= \overline{m}^{-1}(T) t \int_0^t \int_0^s m(a) da f(s, x(s)) ds - \\
& -\overline{m}^{-1}(T) \int_0^t s \int_0^s m(a) da f(s, x(s)) ds - \overline{m}^{-1}(T) \int_0^t s \int_s^T m(a) da f(s, x(s)) ds - \\
& -\overline{m}^{-1}(T) t \int_t^T \int_s^T m(a) da f(s, x(s)) ds
\end{aligned}$$

we obtain (4).

References

- [1] A.D.Polyanin, V.F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*. Chapman & Hall/CRC, Boca Raton (2004).
- [2] L. Zheng, X. Zhang, *Modeling and Analysis of Modern Fluid Problems*. Mathematics in Science and Engineering. Academic Press, London (2017).

Convection and Constitutive Memory: An Evolutionary Operator Viewpoint

Brian Straughan

University of Durham, Durham, United Kingdom
(e-mail: brian.straughan@durham.ac.uk)

Recent developments in generalized heat conduction have led to renewed interest in convection problems involving constitutive laws with memory. In contrast to the classical Fourier law, rate-type heat flux models such as the Maxwell–Cattaneo and Jeffreys laws introduce relaxation and retardation effects which fundamentally alter the structure of the governing equations and the nature of convective instability.

This lecture discusses thermal convection in clear fluids and porous media when constitutive memory effects are present in the heat flux law. Particular attention is given to Jeffreys-type heat models and their relationship to the classical theories of Cattaneo and to higher-order constitutive hierarchies. It is shown that the relative sizes of the relaxation and retardation parameters play a crucial role in determining the character of the instability, including conditions under which oscillatory convection may or may not occur.

A key theme of the lecture is the relationship between generalized heat conduction and the evolutionary operator approach developed in the work of Anatoly P. Oskolkov and the Ladyzhenskaya school. In this setting, constitutive laws may be viewed as operator relations between heat flux and temperature gradient, analogous to corresponding hierarchies in viscoelastic fluid mechanics. This viewpoint provides a unified framework linking convection, constitutive memory, and dissipative evolutionary systems.

Specific applications to Bénard convection in a Darcy porous model are discussed, employing a Jeffreys–Oskolkov heat model or a Kelvin–Voigt–Oskolkov heat theory.

On the Optimal Control Problem of a Second-Order Elliptic Equation with Periodic Boundary Conditions

Rafiq Tagiyev, Aytaj Mammadova

Baku State University, Baku, Azerbaijan

(e-mail: r.tagiyev@list.ru, amamedova9020@gmail.com)

Optimal control problems governed by elliptic equations arise in numerous areas of science and engineering, including heat conduction, diffusion, electrostatics, and other phenomena [1,2]. Although optimal control problems governed by elliptic equations with classical boundary conditions have been extensively studied, comparatively less attention has been devoted to problems with periodic boundary conditions [3]. Such problems provide mathematical models for processes exhibiting periodic behaviour in the spatial variables and have applications in various areas of physics and engineering.

1. Statement of the problem.

Let $\Omega = \{x = (x_1, x_2) \in E^2 : 0 < x_i < l_i, i = 1, 2\}$ be a rectangular domain. Consider the controlled process governed by the boundary value problem:

$$-\sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(k_i(x) \frac{\partial u}{\partial x_i} \right) + q(x)u = f(x), \quad x \in \Omega, \quad (1)$$

$$u|_{x_i=0} = u|_{x_i=l_i}, \quad i = 1, 2, \quad (2)$$

$$k_i(x) \frac{\partial u}{\partial x_i} \Big|_{x_i=0} = k_i(x) \frac{\partial u}{\partial x_i} \Big|_{x_i=l_i}, \quad i = 1, 2. \quad (3)$$

Here $v(x) = (q(x), f(x))$ is a control, the state function $u = u(x, v)$ is the solution to the boundary value problem (1)-(3) corresponding to the control $v = v(x)$, and $k_i(x)$, $i = 1, 2$ is a given measurable function.

We introduce the admissible control set

$$V = \{v(x) = (q(x), f(x)) \in H = L_2(\Omega) \times L_2(\Omega) \equiv (L_2(\Omega))^2 :$$

$$0 \leq q_0 \leq q(x) \leq q_1, \quad f_0 \leq f(x) \leq f_1, \quad \text{a.e. in } \Omega\}. \quad (4)$$

Here q_0, q_1, f_0, f_1 are given numbers.

The objective is to minimize the functional

$$J(v) = \int_{\Omega} |u(x, v) - z(x)|^2 dx \quad (5)$$

subject to the conditions (1)-(3), here $z(x) \in L_2(\Omega)$ is a given function.

Assume that $k_i(x)$, $i = 1, 2$, satisfy the following conditions:

$$0 < \nu_i \leq k_i(x) \leq \mu_i, \quad \text{a.e. } x \in \Omega, \quad \nu_i, \mu_i = \text{const} > 0, \quad i = 1, 2.$$

For a fixed control $v = v(x) \in V$, a function $u = u(x, v) \in \hat{W}_2^1(\Omega) = \{u \in W_2^1(\Omega) : u|_{x_i=0} = u|_{x_i=l_i}, i = 1, 2\}$ is called a generalized solution to the boundary value problem (1)-(3) in $W_2^1(\Omega)$ if, for every $\eta = \eta(x) \in \hat{W}_2^1(\Omega)$, the following integral identity holds:

$$\int_{\Omega} \left(\sum_{i=1}^2 k_i(x) \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_i} + q(x) u \eta \right) dx = \int_{\Omega} f(x) \eta dx.$$

The following theorem proves the well-posedness of problem (1)-(5).

Theorem 1. Assume that the assumptions of problem (1)-(5) are satisfied. Then the functional $J(v)$ is weakly continuous on V , and the set $V_* = \{v_* \in V : J(v_*) = J_* \equiv \inf\{J(v) : v \in V\}\}$ is nonempty and weakly compact in H . Moreover, every minimizing sequence $\{v_k(x)\} \subset V$ for the functional (5) converges weakly to the set V_* in H .

It follows from Theorem 1 that problem (1)-(5) has a solution. However, as the following example shows, the solution to problem (1)-(5) is not necessarily unique.

Example 1. Let $\Omega = (0, 1) \times (0, 1)$, $k_1(x) = k_2(x) = 1$, $\nu_1 = \nu_2 = 1$, $\mu_1 = \mu_2 = 2$, $q_0 = 0.5$, $q_1 = 3$, $f_1 = -8\pi^2 - 2$, $f_2 = 8\pi^2 + 2$, and the function $z(x) = \sin(2\pi x_1) \sin(2\pi x_2)$. Then one can readily see that the minimum value of the functional $J(v)$ is attained at two controls

$$v_{1*}(x) = (q_{1*}(x), f_{1*}(x)) = (1, (8\pi^2 + 1) \sin(2\pi x_1) \sin(2\pi x_2)),$$

$$v_{2*}(x) = (q_{2*}(x), f_{2*}(x)) = (2, (8\pi^2 + 2) \sin(2\pi x_1) \sin(2\pi x_2)),$$

and $u(x, v_1) = u(x, v_2) = \sin(2\pi x_1) \sin(2\pi x_2) = z$, therefore $J(v_{1*}) = J(v_{2*}) = J_* = 0$. Hence, the solution to problem (1)-(5) is not unique.

The following example shows that a minimizing sequence for the functional $J(v)$ does not necessarily have a limit in H .

Example 2. Let $\Omega = (0, 1) \times (0, 1)$, $k_1(x) = k_2(x) = 1$, $q_0 = 0, 5\pi^2$, $q_1 = \pi^2 + 1$, $f_1 = 1, 5\pi^2$, $f_2 = 3\pi^2$, and $z(x) = \sin(\pi x_1) \sin(\pi x_2)$. Then $v_* = (\pi^2, 3\pi^2 \sin(\pi x_1) \sin(\pi x_2))$ is an optimal control, and $u(x, v_*) = \sin(\pi x_1) \sin(\pi x_2)$, $x \in \Omega$, $J(v_*) = J_* = 0$. Take the sequence of controls $v_m(x) = (\pi^2 + \sin(\pi m x_1), (3\pi^2 + \sin(\pi m x_1)) \sin(\pi x_1) \sin(\pi x_2)) \in V$, ($m = 1, 2, \dots$), $x \in \Omega$. Then $v_m(x)$ converges weakly to $v_*(x)$ in H . Therefore, $J(v_m) \rightarrow J(v_*) = J_* = 0$. Hence, $\{v_m\}$ is a minimizing sequence for the functional $J(v)$. However, this sequence does not have a limit in H , since $\{\sin(\pi m x_1)\}$ is not strongly convergent in $L_2(\Omega)$.

We now introduce the adjoint problem associated with problem (1)-(5):

$$-\sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(k_i(x) \frac{\partial \psi}{\partial x_i} \right) + q(x) \psi = 2[z(x) - u(x, v)], \quad x \in \Omega, \quad (6)$$

$$\psi|_{x_i=0} = \psi|_{x_i=l_i}, \quad i = 1, 2, \quad (7)$$

$$k_i(x) \frac{\partial \psi}{\partial x_i} \Big|_{x_i=0} = k_i(x) \frac{\partial \psi}{\partial x_i} \Big|_{x_i=l_i}, \quad i = 1, 2. \quad (8)$$

A solution to the boundary value problem (6)-(8) corresponding to a fixed control $v = v(x) \in V$ is understood as a generalized solution in $W_2^1(\Omega)$.

Define the Hamilton–Pontryagin function for problem (1)-(5) by

$$H(u, q, f, \psi) = -q u \psi + f \psi.$$

Theorem 2. Let $v_*(x) = (q_*(x), f_*(x)) \in V$ be an optimal control for problem (1)-(5), and $u_*(x) = u(x, v_*)$, $\psi_*(x) = \psi(x, v_*)$ denote the solutions to problems (1)-(3) and (6)-(8), respectively, corresponding to the control $v_*(x)$. Then the control $v_*(x) = (q_*(x), f_*(x)) \in V$ satisfies the maximum condition:

$$\max_{\substack{q_0 \leq q \leq q_1 \\ f_0 \leq f \leq f_1}} H(u_*(x), q, f, \psi_*(x)) = H(u_*(x), q_*(x), f_*(x), \psi_*(x)), \quad x \in \Omega. \quad (9)$$

Remark. If $q(x)$ is a given function in problem (1)–(5), then the maximum condition (9) is also a sufficient condition for the optimality of the control $f_*(x)$.

References

- [1] K.A. Lur'e, Optimal Control in Problems of Mathematical Physics, Moscow: Nauka, 1975.
- [2] V.G. Litvinov, Optimization in Elliptic Boundary Value Problems with Applications to Mechanics, Moscow: Nauka, 1987.
- [3] R.K. Tagiyev, A.K. Mammadova, An Optimal Control Problem for an Elliptic Equation with Periodic Boundary Conditions, Baku State University Journal of Mathematics & Computer Science, 2024, 1(4), 39-48.

On the Representation of Solutions to a Linear Hyperbolic Equation with Fractional Partial Derivatives in the Caputo Sense

Shakir Yusubov^{a,b}, Elimhan Mahmudov^{c,d},
Shikhi Yusubov^e

^a*Baku State University, Baku, Azerbaijan*

^b*Institute of Mathematics, Baku, Azerbaijan*

^c*Azerbaijan National Aviation Academy, Baku, Azerbaijan*

^d*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*

^e*Shanghai University, Shanghai, China*

(e-mail: yusubov_shakir@mail.ru, elimhan22@yahoo.com,
yusubovsixi@gmail.com)

Introduction. It should be noted that in certain optimal control problems described by fractional-order hyperbolic partial differential equations, the boundary conditions are given by linear fractional-order ordinary differential equations. In optimal control problems of this kind, obtaining high-order necessary optimality conditions for singular controls requires the use of an integral representation of the solution to the corresponding linearized problem. Therefore, the purpose of this work is to derive an integral representation of the solution to a problem governed by a linear fractional-order hyperbolic partial differential equation, subject to boundary conditions specified by linear fractional-order ordinary differential equations.

First, the existence and uniqueness of the solution to the problem under consideration in the class $AC_\infty^\alpha(G, \mathbb{R}^n)$ is proved. Subsequently, the equation and the boundary conditions are treated together as components of a single operator equation. Consequently, the fundamental solution of the considered problem is determined as the solution to a certain system of matrix equations for every fixed $x \in G$. The fundamental solution consists of four matrix components, each of which is a solution to the corresponding equation of the system under consideration. One of the components is the principal part of the fundamental solution. The remaining components of the fundamental solution are expressed in terms of it. The integral representation of the solution to the problem under consideration is expressed in terms of the fundamental solution.

Problem Statement and Obtained Result. Let us consider the following nonhomogeneous problem for the linear hyperbolic partial differential equation with variable coefficients involving the Caputo fractional derivatives of the order $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \in (0, 1)$, $i = 1, 2$:

$$(D_1^{\alpha_1} D_2^{\alpha_2} z)(x) = c(x)z(x) + f(x), \quad (1)$$

$$\text{a.e. } x \in G = G_1 \times G_2 = [x_1^0, X_1] \times [x_2^0, X_2],$$

with the following conditions

$$(D_1^{\alpha_1} z)(x_1, x_2^0) = a_1(x_1)z(x_1, x_2^0) + f_1(x_1), \quad \text{a.e. } x_1 \in G_1, \quad (2)$$

$$(D_2^{\alpha_2} z)(x_1^0, x_2) = a_2(x_2)z(x_1^0, x_2) + f_2(x_2), \quad \text{a.e. } x_2 \in G_2, \quad (3)$$

$$z(x_1^0, x_2^0) = f_0. \quad (4)$$

Here $D_1^{\alpha_1} D_2^{\alpha_2}$ is the mixed Caputo fractional derivative of order $\alpha = (\alpha_1, \alpha_2) \in (0, 1) \times (0, 1)$, $D_i^{\alpha_i}$, $i = 1, 2$ are the partial α_i order Caputo derivatives [1], $z(x) \in \mathbb{R}^n$ is the sought function, $c(\cdot) \in L^\infty(G, \mathbb{R}^{n \times n})$, $a_i(\cdot) \in L^\infty(G_i, \mathbb{R}^{n \times n})$, $i = 1, 2$, $f(\cdot) \in L^\infty(G, \mathbb{R}^n)$, $f_i(\cdot) \in L^\infty(G_i, \mathbb{R}^n)$, $i = 1, 2$, $f_0 \in \mathbb{R}^n$; $x_i^0 < X_i$, $i = 1, 2$, $x_i^0, X_i \in \mathbb{R}$, $i = 1, 2$ are fixed.

By $AC_\infty^\alpha(G, \mathbb{R}^n)$ [3] we denote the set of all functions $z(\cdot) : G \rightarrow \mathbb{R}^n$, such that

$$z(x_1, x_2) = \varphi_0 + (I_{x_1^0+}^{\alpha_1} \varphi_1)(x_1) + (I_{x_2^0+}^{\alpha_2} \varphi_2)(x_2) + (I_{x^0+}^\alpha \varphi)(x),$$

$$x = (x_1, x_2) \in G,$$

with $\varphi_0 \in \mathbb{R}^n$, $\varphi_1(\cdot) \in L^\infty(G_1, \mathbb{R}^n)$, $\varphi_2(\cdot) \in L^\infty(G_2, \mathbb{R}^n)$, and $\varphi(\cdot) \in L^\infty(G, \mathbb{R}^n)$. Here, $I_{x_1^+}^{\alpha_1}$, $I_{x_2^+}^{\alpha_2}$, and $I_{x^+}^\alpha$ are the left-sided Riemann-Liouville fractional integral operators [2].

A function $z : G \rightarrow \mathbb{R}^n$ is called a solution of problem (1)-(4), if the inclusion $z(\cdot) \in AC_\infty^\alpha(G, \mathbb{R}^n)$ is valid, initial condition (4) holds, and differential equations (1), (2), (3) are satisfied for almost every $x \in G$, $x_1 \in G_1$, $x_2 \in G_2$, respectively. Note that, for $a_1(x_1) \equiv a_2(x_2) \equiv 0$, $x_1 \in G_1$, $x_2 \in G_2$, the existence and uniqueness of the solution to problem (1)-(4) were proved in paper [3]. The following theorem establishes the existence and uniqueness of the solution to problem (1)-(4) for $a_1(x_1) \neq 0$, $x_1 \in G_1$, $a_2(x_2) \neq 0$, $x_2 \in G_2$.

We have proved the following theorems:

Theorem 1. *Problem (1)-(4) has a unique solution in the space $AC_\infty^\alpha(G, \mathbb{R}^n)$.*

Theorem 2. *Let $c(\cdot) \in L^\infty(G, \mathbb{R}^{n \times n})$, $f(\cdot) \in L^\infty(G, \mathbb{R}^n)$, $a_i(\cdot) \in L^\infty(G_i, \mathbb{R}^{n \times n})$, $i = 1, 2$, $f_0 \in \mathbb{R}^n$. Then the solution z of problem (1)-(4) can be represented in the form*

$$\begin{aligned} z(x) = & \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1^0}^{x_1} \int_{x_2^0}^{x_2} (x_1 - t_1)^{\alpha_1-1} (x_2 - t_2)^{\alpha_2-1} F(x_1, x_2; t_1, t_2) \\ & \times f(t_1, t_2) dt_2 dt_1 + \frac{1}{\Gamma(\alpha_1)} \int_{x_1^0}^{x_1} (x_1 - t_1)^{\alpha_1-1} F_1(x_1, x_2; t_1) f_1(t_1) dt_1 \\ & + \frac{1}{\Gamma(\alpha_2)} \int_{x_2^0}^{x_2} (x_2 - t_2)^{\alpha_2-1} F_2(x_1, x_2; t_2) f_2(t_2) dt_2 + F_0(x_1, x_2) f_0, \quad x \in G. \end{aligned}$$

Here, $F(x; t)$, $F_1(x; t_1)$, $F_2(x; t_2)$, and $F_0(x)$ are solutions to certain matrix equations.

References

- [1] S. Abbas, M. Benchohra, A.N. Vityuk, *On fractional order derivatives and Darboux problem for implicit differential equations*, *Fract. Calc. Appl. Anal.*, **15** (2012), 168-182.
- [2] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.

- [3] Sh.Sh. Yusubov, E.N. Mahmudov, *Necessary and sufficient optimality conditions for fractional Fornasini-Marchesini model*, J. Indust. Manag. Optim. **19**(10) (2023), 7221-7244.

Sharp Bounds for the Attractors Dimension for 2D Navier-Stokes Equations

Sergey Zelik

Zhejiang Normal University, Zhejiang, China
(e-mail: s.zelik@surrey.ac.uk)

We present a principally new method for obtaining the lower bounds for the attractors dimensions of the equations related with hydrodynamics, which is not based on the Kolmogorov flows, and apply it to the classical 2D Navier–Stokes equations in a bounded domain as well as for the Navier–Stokes equations with Ekman damping in the whole plane. In particular, in the case of bounded domains, we give the lower bounds, which are similar to the well-known estimate on a torus. In both cases our estimates are sharp. Note that no lower bounds for these two cases were known before. We suggest to use the so-called multi-vortex, which consists of a well- separated Vishik vortices (i.e., spectrally unstable localized in space flows constructed by M.M. Vishik), as the analogue of the Kolmogorov flows. Note also that this method reproduces the known result on the torus and that it is applicable to many other equations of hydrodynamics

FUNCTIONAL ANALYSIS AND THEORY OF OPERATORS**On the Absolute and Uniform Convergence of The Expansion in the Eigenfunctions one of a Boundary Value Problem for a Third-Order Differential Equation****Yuliya Abbasova***Azerbaijan State Pedagogical University, Baku, Azerbaijan*
(e-mail: abbasovayuliya88@gmail.com)

We consider the spectral problem

$$\begin{aligned} y''' - y' + \lambda y &= 0, \quad x \in (0, 1) \\ y''(0) - y'(0) - y(1) &= 0 \\ y''(1) + y'(1) + y'(0) &= 0 \\ y'(1) - y'(0) + 2y(0) + 2y(1) &= 0 \end{aligned} \tag{1}$$

The eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of problem (1) are located on the imaginary axis, the eigenfunctions $\{u_n(x)\}_{n=1}^{\infty}$ form a complete orthonormal system in $L_2(0, 1)$.

Let us introduce the notation $\lambda = \rho^3$, $S_0 = \{\rho : 0 \leq \arg \rho \leq \frac{\pi}{3}\}$, $S_1 = \{\rho : \frac{\pi}{3} \leq \arg \rho \leq \frac{2\pi}{3}\}$, $\lambda = \rho_n^3$. Sequence of numbers $\{\rho_n\}$ consists of two series

$$\begin{aligned} \rho_n^{(1)} &= \left(2n + \frac{1}{3}\right) \pi i + O(n^{-1}), \quad n = N_0, N_0 + 1, \dots; \\ \rho_n^{(2)} &= e^{-\pi i/3} \left[\left(2n + \frac{1}{3}\right) \pi i + O(n^{-1}) \right], \quad n = N_0, N_0 + 1, \dots, \end{aligned}$$

where N_0 - a sufficiently large natural number.

The corresponding non-normalized eigenfunctions have the following asymptotics:

$$\begin{aligned} u_n^{(1)}(x) &= \frac{i}{\sqrt{3}} e^{\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)(2n+\frac{1}{3})\pi x} + \\ &+ \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) e^{-i(2n+\frac{1}{3})\pi x} + e^{\left(\frac{\sqrt{3}}{2} + \frac{i}{3}\right)(2n+\frac{1}{3})\pi(x-1)} + O(n^{-1}), \end{aligned}$$

$$u_n^{(2)}(x) = -e^{\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)(2n+\frac{1}{3})\pi x} + \\ + \left(\frac{i\sqrt{3}}{2} - \frac{1}{2}\right) e^{i(2n+\frac{1}{3})\pi x} + e^{\left(\frac{\sqrt{3}}{2}-\frac{i}{3}\right)(2n+\frac{1}{3})\pi(x-1)} + O(n^{-1}).$$

Let us denote by $\sigma_\nu(x, f) = \sum_{|\rho_k| \leq \nu} (f, u_k) u_k(x)$, partial sum of order $\nu, \nu > 0$, spectral expansion of the function $f(x) \in W_p^1(G)$, $p > 1$, by the system $\{u_n(x)\}$ and introduce the difference $R_\nu(x, f) = f(x) - \sigma_\nu(x, f)$.

In this paper the following theorem is proved.

Theorem 1. Spectral expansion of an arbitrary function $f(x) \in W_p^1(G)$, $p > 1$, converges absolutely and uniformly on $\overline{G}$ and the following estimate is true:

$$\max_{x \in \overline{G}} |R_\nu(x, f)| = o(\nu^{-\beta}), \quad \nu \rightarrow \infty,$$

where $\beta = \min \left\{ \frac{1}{q}, \frac{1}{2} \right\}$.

Corollary. If in Theorem 1 the function $f(x) \in W_p^1(G)$, $p > 1$, satisfy the condition $f(0) = f(1) = 0$, then

$$\max_{x \in \overline{G}} |R_\nu(x, f)| \leq \text{const } \nu^{-\beta} \|f'\|_p, \quad \nu \geq 2,$$

where const does not depend on $f(x)$ and ν .

References

- [1] V.M. Kurbanov, Yu. G. Abbasova, Convergence of the spectral decomposition of functions from the class $W_{p,m}^1(G)$, $p > 1$, in the vector eigen functions of a differential operator of the third order. *Ukr.Mat.Zh.*, 69(6), 2017, 719-733.

Boundary Value Problems for a Class of Fourth-Order Operator-Differential Equations on a Finite Interval

Gulsum Aghayeva^{a,b}, Davron Juraev^b, Nurbala Suleymanov^c

^a*Baku State University, Baku, Azerbaijan*

^b*University, of Economics and Pedagogy, Karshi, Uzbekistan*

^c*Guba branch of Azerbaijan State Pedagogical University, Guba, Azerbaijan*

(e-mail: gulsumm.agayeva@mail.ru, juraevdavron12@gmail.com,
nurbalanurbala210@gmail.com)

Introduction. Consider the following boundary value problem in a separable Hilbert space H :

$$\left(\frac{d^2}{dt^2} - A^2\right)^2 u(t) + \sum_{j=0}^4 A_{4-j} u^{(j)}(t) = f(t), \quad t \in (0, 1), \quad (1)$$

$$u(0) = u''(0) = 0, \quad u(1) = u''(1) = 0, \quad (2)$$

where $f(t)$, $u(t)$ are H -valued functions defined almost everywhere on the interval $(0,1)$.

The operator coefficients of equation (1) satisfy the following assumptions:

1) A is self-adjoint, positive definite operator in H .

2) $B_j = A_j A^{-j}$ ($j = \overline{0,4}$) are bounded operators in H .

Let $L_2((0,1):H)$ denote the Hilbert space of square-integrable H -valued functions equipped with the norm

$$\|f\|_{L_2((0,1):H)} = \left(\int_0^1 \|f(t)\|^2 dt\right)^{\frac{1}{2}} < \infty.$$

Following [1] we introduce the Sobolev-type spaces

$$W_2^4((0,1):H) = \{u : u^{(4)} \in L_2((0,1):H), A^4 u \in L_2((0,1):H)\}$$

equipped with the norm

$$\|u\|_{W_2^4((0,T):H)} = \left(\|u^{(4)}\|_{L_2((0,1):H)}^2 + \|A^4 u\|_{L_2((0,1):H)}^2\right)^{\frac{1}{2}}$$

This space serves as the functional framework for the investigation of the boundary value problem and the analysis of its regular solvability.

Main results: A notion of **regular solvability** is introduced for the boundary value problem in the Sobolev-type space $W_2^4((0, 1) : H)$. Under this definition, every function $f \in L_2((0, 1); H)$ in $f \in L_2((0, 1) : H)$ admits a unique regular solution satisfying the boundary conditions in the trace sense together with the corresponding a priori estimate.

Furthermore, the operators

$$P_0 u = \left(\frac{d^2}{dt^2} - A^2 \right)^2 u, \quad P_1 u = \sum_{j=0}^4 A_{4-j} u^{(j)}, \quad W_2^4((0, 1) : H) \in L_2((0, 1) : H).$$

are introduced, and the operator

$$P = P_0 + P_1$$

Theorem 1. Assume that the conditions (1) and (2) are satisfied.

Then the operator

$$P_0 : W_2^4((0, 1) : H) \rightarrow L_2((0, 1) : H)$$

is an isomorphism.

Theorem 2. Assume that the conditions (1) and (2) hold. Then, for every

$$u \in W_2^4((0, 1) : H)$$

the following estimate is valid:

$$\|A^{4-j} u^{(j)}\|_{L_2((0,1);H)} \leq d_{4,j}^2 \|P_0 u\|_{L_2((0,1);H)}, \quad j = \overline{0, 4}$$

where $d_{4,0} = d_{4,4} = 1$, and

$$d_{4,j} = \left(\frac{j}{4} \right)^{\frac{j}{4}} \left(\frac{4-j}{4} \right)^{\frac{4-j}{4}}, \quad j = 1, 2, 3$$

Theorem 3. Assume that the conditions (1) and (2) are satisfied. Furthermore, suppose that

$$q = \sum_{j=0}^4 d_{4,j}^2 \|B_j\| < 1,$$

Then the boundary value problem (1)-(2) is regularly solvable.

Conclusion

In this study, a boundary value problem for a class of fourth-order operator-differential equations in a separable Hilbert space was investigated. A functional framework based on Sobolev-type spaces was established, and the concept of regular solvability was introduced.

The associated operator was analyzed, and it was proved that the principal operator defines an isomorphism between the corresponding Hilbert spaces. Furthermore, a priori estimates for the intermediate derivatives of the solution were obtained. Finally, sufficient conditions ensuring the regular solvability of the boundary value problem were established.

The obtained results contribute to the theory of higher-order operator-differential equations and provide a theoretical basis for further investigations of boundary value problems in Hilbert spaces.

References

- [1] J.-L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Springer, 1972.
- [2] S.S. Mirzoyev, *On solvability of a boundary value problem of fourth order in Hilbert spaces*, *Applied Mathematical Sciences*, Vol. 9, No. 128, 2015, pp. 6391–6395.
- [3] S.S. Mirzoyev, *On the norms of operator of intermediate derivatives*, *Transactions of NAS of Azerbaijan*, Vol. XXIII, No. 1, 2003, pp. 157–164.
- [4] S.S. Mirzoyev, *On generalized solutions of a boundary value problem for operator-differential equations on a finite interval*, *Applied Mathematical Sciences*, 11(29), 2017, 1405-1411.
- [5] Mirzoev S.S., Aghayeva G.A. On a Boundary Value Problem for Operator-Differential Equations in Hilbert Space// *Azerbaijan Journal of Mathematics* V. 14,2024, January ,p 109-121
- [6] Aliyev A.R. Boundary Value Problems for a Class of High-Order Operator-Differential Equations with Variable Coefficients. *Matem zametki*, 74(6), 803-814 (2003).

Global Spectral Configurations of a Two-Particle Lattice Schrödinger Operator

Shakhnoza Akhmatova

Samarkand State University, 140104, Samarkand, Uzbekistan
(e-mail: mukhayyo.akhmadova@mail.ru, shakhnoza_a1@mail.ru)

Discrete Schrödinger operators on lattices form an important class of models in mathematical physics and provide a natural framework for describing quantum systems in periodic structures, including crystalline solids and optical lattices [1,2]. Owing to translational invariance, such Hamiltonians can be analyzed by means of the Floquet–Bloch decomposition, which represents the full operator as a direct integral of fiber operators indexed by the total quasi-momentum [2,3].

Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \equiv (-\pi, \pi]$ denote the one-dimensional torus, and let $L^{2,e}(\mathbb{T})$ be the subspace of even square-integrable functions with respect to the normalized Lebesgue measure $d\nu(t) = dt/(2\pi)$. For each total quasi-momentum $K \in \mathbb{T}$, the fiber Hamiltonian $H_{\boldsymbol{\mu}}(K)$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$, acts in $L^{2,e}(\mathbb{T})$ as

$$\begin{aligned} H_{\boldsymbol{\mu}}(K) &= H_0(K) + V_{\boldsymbol{\mu}}, \\ (H_0(K)f)(p) &= \mathcal{E}_K(p)f(p), \quad \mathcal{E}_K(p) = 4 \left(1 - \cos \frac{K}{2} \cos p \right), \\ (V_{\boldsymbol{\mu}}f)(p) &= \sum_{i=1}^3 \mu_i \langle \phi_i, f \rangle \phi_i(p), \\ \phi_1(t) &= 1, \quad \phi_2(t) = \sqrt{2} \cos t, \\ \phi_3(t) &= \sqrt{2} \cos 2t, \quad t \in \mathbb{T}. \end{aligned} \tag{1}$$

The form factors in (1) are orthonormal. The free fiber Hamiltonian $H_0(K)$ has purely absolutely continuous spectrum consisting of a single band:

$$\begin{aligned} \sigma(H_0(K)) &= [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)], \\ \mathcal{E}_{\min}(K) &= 4 \left(1 - \cos \frac{K}{2} \right), \quad \mathcal{E}_{\max}(K) = 4 \left(1 + \cos \frac{K}{2} \right). \end{aligned} \tag{2}$$

Since V_μ is a finite-rank operator, the essential spectrum of $H_\mu(K)$ is

$$\sigma_{\text{ess}}(H_\mu(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]. \quad (3)$$

Let $n_-(H_\mu(K))$ and $n_+(H_\mu(K))$ denote the numbers of discrete eigenvalues, counting multiplicities, below and above the essential spectrum, respectively. Because the perturbation in (1) has rank at most three,

$$n_-(H_\mu(K)) + n_+(H_\mu(K)) \leq 3, \quad K \in \mathbb{T}. \quad (4)$$

At zero total quasi-momentum, $K = 0$, the essential spectrum is the interval $[0, 8]$. We treat μ_3 as a fixed structural parameter and analyze the attainable spectral states as the remaining coupling intensities (μ_1, μ_2) vary over the entire real plane. For fixed $\mu_3 \in \mathbb{R}$, define the global spectral configuration mapping

$$N_{\mu_3} : \mathbb{R}^2 \rightarrow \mathbb{Z}_{\geq 0}^2, \quad N_{\mu_3}(\mu_1, \mu_2) = (n_-(H_\mu(0)), n_+(H_\mu(0))), \quad (5)$$

and the set of all reachable configurations

$$S_{\mu_3} := \text{Im}(N_{\mu_3}) = \{N_{\mu_3}(\mu_1, \mu_2) : (\mu_1, \mu_2) \in \mathbb{R}^2\}. \quad (6)$$

The global classification is given by the following theorem.

Theorem 1. *The set of reachable spectral configurations S_{μ_3} is uniquely determined by the value of μ_3 as follows:*

(i) *If $\mu_3 < -2$, then*

$$S_{\mu_3} = \{(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (3, 0)\}.$$

(ii) *If $|\mu_3| \leq 2$, then*

$$S_{\mu_3} = \{(0, 0), (0, 1), (0, 2), (1, 0), \\ (1, 1), (1, 2), (2, 0), (2, 1)\}.$$

(iii) *If $\mu_3 > 2$, then*

$$S_{\mu_3} = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (2, 1)\}.$$

Therefore, the parameter μ_3 determines the global structure of the attainable spectral configurations, while the rank-three constraint (4) rules out all pairs (n_-, n_+) satisfying $n_- + n_+ > 3$.

References

- [1] D. Mattis, *The few-body problem on a lattice*, Rev. Mod. Phys. **58** (1986), 361–379.
- [2] S. Albeverio, S.N. Lakaev, K.A. Makarov, Z.I. Muminov, *The threshold effects for the two-particle Hamiltonians on lattices*, Comm. Math. Phys. **262** (2006), 91–115.
- [3] S.N. Lakaev, M.O. Akhmadova, *The Number and Location of Eigenvalues for the Two-Particle Schrödinger Operators on Lattices*, Complex Anal. Oper. Theory **17** (2023), Article 92.

Self-Adjointness of the Schrödinger Operator With Slowly Decaying Potential

Araz Aliev^{a,c}, Elshad Eyvazov^{b,c}

^aAzerbaijan State Oil and Industry University, Baku, Azerbaijan

^bBaku State University, Baku, Azerbaijan

^cInstitute of Mathematics, Baku, Azerbaijan

(e-mail: alievaraz@yahoo.com, eyvazoveshad@gmail.com)

As is well known, many important problems in quantum mechanics can be reduced to the spectral analysis of a second-order elliptic partial differential operator of the form

$$(H\psi)(x) = -\frac{\hbar^2}{2m}(\Delta\psi)(x) + (V\psi)(x) \quad (1)$$

known as the Schrödinger operator. For example, the Hamiltonian for an electron bound to a nucleus of charge Ze can be represented in the form (1) with the electric potential $V(x) = -\frac{Ze^2}{|x|}$ where $e \approx 1,6 \times 10^{-9}C$ is the elementary charge.

Historically, the stability of the Coulomb problem is probably the most important achievement of quantum mechanics. However, we would like to emphasize that this is not a special case.

Note that the self-adjointness and lower boundedness of the Hamiltonian are important in quantum mechanics. Self-adjointness is necessary for the proper definition of the dynamics, while lower boundedness ensures the stability of the system. It is known (see [1]) that for $\alpha > 2$, the Schrödinger operator $H := -\Delta - \frac{Ze^2}{|x|^\alpha}$, defined at least for functions in $C_c^\infty(R^d \setminus \{0\})$ whose support does not include the origin, is unbounded from below. Consequently, all standard theorems on quadratic forms are inapplicable. When $\alpha > 2$, quantum particles approaching from infinity are “absorbed into the hole” at the origin, moving faster and faster as they approach the origin.

Our goal is to prove the self-adjointness and lower boundedness of the Hamiltonian $H := -\Delta - \frac{Ze^2}{|x|^\alpha}$ in the space $L^2(R^d)$ for $d \geq 3$ and $0 \leq \alpha < 2$.

Let v_α be the sesquilinear form generated by the potential $V_\alpha(x) = -\frac{Ze^2}{|x|^\alpha}$ in $L^2(R^d)$, defined as follows:

$$D(v_\alpha) = \left\{ \psi(x) \in L^2(R^d) : \int_{R^d} \left| -\frac{Ze^2}{|x|^\alpha} \right| |\psi(x)|^2 dx < +\infty \right\},$$

$$v_\alpha(\varphi, \psi) = \int_{R^d} \left(-\frac{Ze^2}{|x|^\alpha} \right) \varphi(x) \overline{\psi(x)} dx, \quad \varphi, \psi \in D(v_\alpha).$$

Lemma 1. *The sesquilinear form v_α is closed, and it corresponds to the maximal multiplication operator by the function $V_\alpha(x) = -\frac{Ze^2}{|x|^\alpha}$ on $L^2(R^d)$.*

Lemma 2. *Let $V_n(x) = -\min \left\{ n, \frac{Ze^2}{|x|^\alpha} \right\}$ and $\psi \in D(v_\alpha)$. Then*

$$v_\alpha(\psi) = \lim_{n \rightarrow \infty} \int_{R^d} V_n(x) |\psi(x)|^2 dx.$$

Let h_0 be the quadratic form generated by the Laplace operator, i.e.,

$$D(h_0) = H^1(R^d) \quad h_0(\psi) = \sum_{k=1}^d \int_{R^d} \left| \frac{\partial \psi}{\partial x_k} \right|^2 dx, \quad \psi \in D(h_0).$$

Theorem. *The form v_α is bounded with respect to h_0 with relative bound less than one.*

Corollary. It follows from the KLMN theorem (see, for example, [2]) that there exists a unique self-adjoint and lower bounded operator $H := -\Delta + V_\alpha$

associated with the form $h = h_0 + v_\alpha$, which is called the form sum of $-\Delta$ and $V_\alpha(x)$.

Remark. *The theorem in the three-dimensional case was proved in [3], and for $\alpha = 1$ in [4].*

References

- [1] E. B. Davies, *Spectral theory and differential operators*, Cambridge University Press, 1995.
- [2] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators : With Applications to Quantum Mechanics and Global Geometry*, Berlin, Heidelberg : Springer Berlin Heidelberg, 1987.
- [3] M. Reed, B. Simon, *Methods of modern mathematical physics.v.2: Fourier analysis. self-adjointness*, Academic Press New York, San Francisco, London, 1975.
- [4] D. Krejcirik, *Schrödinger operators and their spectra*, Basque Center for Applied Mathematics, Bilbao, 2010

An Oscillation Theorem for Solutions of Nonuniformly Degenerate Second-Order Elliptic–Parabolic Equations

Narmin Amanova^a, Rabil Amanov^b, Togrul Muradov^c

^a*Baku State University, Baku, Azerbaijan*

^b*National Aviation Academy of Azerbaijan, Baku, Azerbaijan*

^c*Institute of Mathematics, Baku, Azerbaijan*

(e-mail: amanova.n.93@gmail.com, rabil.amanov@naa.edu.az, togrulmuradov@gmail.com)

This work considers a class of second-order elliptic–parabolic equations *in non-divergence form with nonuniform power-type degeneracy*. An oscillation theorem for solutions of these equations is established.

Let E_n and R_{n+1} be Euclidean spaces of points $x = (x_1, \dots, x_n)$, $n \geq 1$ and $(x, t) = (x_1, \dots, x_n, t)$, respectively, $\Omega \subset E_n$ be a bounded domain with

the boundary $\partial\Omega$ of class C^2 , $Q_T = \Omega \times (-T, 0)$ be a cylinder of the given height $T > 0$, $\partial Q_T = \partial\Omega \times [-T, 0]$ be a lateral surface of the cylinder Q_T and $\Gamma(Q_T) = \{(x, t) | x \in \Omega, t = -T\} \cup (\partial\Omega \times [-T, 0])$ is a parabolic boundary of Q_T . In Q_T consider an elliptic-parabolic equation

$$Lu = \sum_{i,j=1}^n a_{ij}(x, t)u_{ij} + \varphi(0 - t)u_{tt} - u_t = 0. \quad (1)$$

Here, $\|a_{ij}(x, t)\|$ is a real symmetric matrix, moreover for all $(x, t) \in Q_T$

$$\inf_{Q_T} \sum_{i=1}^n \frac{a_{ii}(x, t)}{\lambda_i(x, t)} = \gamma, \quad (2)$$

$$\sigma \equiv \sup_{Q_T} \left[\sum_{i,j=1}^n \frac{a_{ij}^2(x, t)}{\lambda_i(x, t)\lambda_j(x, t)} \middle/ \left(\sum_{i=1}^n \frac{a_{ii}(x, t)}{\lambda_i(x, t)} \right)^2 \right] - \frac{1}{n - e^2} < 0, \quad (3)$$

$$\varphi(z) \in C^1[-T, 0], \varphi(z) \geq 0, \varphi'(z) \geq 0,$$

$$\varphi(0) = 0, \varphi'(0) = 0, \varphi(z) \geq \beta_1 z \cdot \varphi'(z), \beta_1 > 0 - \text{is a constant.} \quad (4)$$

Here $\gamma \in (0, 1]$ is a constant, $u = u(x, t)$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \dots, n$, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $e = \inf_{Q_T} \sum_{i=1}^n \frac{a_{ii}(x, t)}{\lambda_i(x, t)} \middle/ \sup_{Q_T} \sum_{i=1}^n \frac{a_{ii}(x, t)}{\lambda_i(x, t)}$, $\lambda_i(x, t) = \left[\frac{\omega_i^{-1}(\rho(x) + \sqrt{|t|})}{\rho(x) + \sqrt{|t|}} \right]^2$, $i = 1, \dots, n$.

The following theorem is true.

Theorem 1. *Let a solution $u(x, t)$ of the equation (1) be defined in the region D , whose coefficients satisfy the conditions (2)-(4). Then for any $\rho > 0$ there exist constants $\lambda(\gamma, n)$, $C_9(\gamma, n, \rho)$ such that*

$$\|u\|_{C_\lambda(D^\rho)} \leq C_9 \cdot \|u\|_{C(D)}.$$

References

- [1] R.A. Amanov, N.R. Amanova, *On solvability of some quasilinear parabolic equations of higher order*, Azerbaijan Journal of Mathematics, 2025, 15(2), 240–263.

- [2] N.R. Amanova, A.I. Ismailov, R.A. Amanov, T.R. Muradov, *A Theorem on the Oscillation of Solutions to Nonuniformly Degenerate Second-Order Elliptic-Parabolic Equations*, Journal of Contemporary Applied Mathematics, 2026, 16(2), 148-168.

Investigation of Resolvent of Operator Generated by Differential Expressions With Operator Coefficients in Hilbert space

Hamidulla Aslanov^a, Gunel Eyvazli^b, Kamilla Alimardanova^a

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan Technical University, Baku, Azerbaijan*

(e-mail: aslanovhamidulla50@gmail.com, gunel.eyvazli@aztu.edu.az,
kamillaalimardanova@gmail.com)

One of the main problems of spectral theory of differential operators is the research of completeness of the system of eigen-and adjoint functions of given operator. The completeness of eigen-and adjoint functions for polynomial operator bundles was researched for the first time by M.V.Keldysh [5]. Later on, many authors dealt with these problems and got fundamental results. In this paper such problems were also researched. The theorems on boundedness of resolvent of operators generated by high order differential expressions with unbounded operator coefficient in Hilbert space were proved. The proved theorems play an important role in research of completeness of the system of eigen-and adjoint elements. In this direction we should note the works of J.E.Allahverdiyev [2], M.G.Gasymov [4], S.S.Mirzoev [6], A.A.Shkalikov [7], Sh. Agmon [1], F. Browder [3] and others.

Let $H_0, H_1, \dots, H_{2m}$ be Hilbert spaces, $H_{i+1} \subset H_i$, $i = 0, 1, \dots, 2m-1$, and also all inclusions are compact.

Let's consider the differential expressions

$$L(x, D)u = \sum_{|\alpha| \leq 2m} A_\alpha(x) D^\alpha u, \quad x \in R^n \quad (1)$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

The function $u(x) \in H_{2m}$ is such that $D^\alpha u \in H_{2m-|\alpha|}$, it assumes that for every $x \in R^n$ operator $A_\alpha(x)$, $\alpha \neq 0$ is bounded: $H_{2m-|\alpha|} \rightarrow H_0$, $A_0(x) = A_0 + \gamma(x)$ where A_0 is a positive defined selfadjoint operator $H_0 \rightarrow H$ such that A_0^{-1} is completely continuous.

Complex -valued function $\gamma(x)$ is assuming to be measurable and local bounded. The theorem on boundedness of resolvent of the operator $L(x, D)$ generated by the expression (1) is proved.

Problem statement. Before the going over the finding the estimate of resolvent of the operator $L(x, D)$ which is main for our purpose, we will formulate the conditions the coefficients of operator $L(x, D)$ must satisfy later.

We will denote

$$R_0(x, \xi) = \left[\sum_{|\alpha|=2m} A_\alpha(x) (i\xi)^\alpha \right]^{-1}, \xi = (\xi_1, \xi_2, \dots, \xi_n), \\ \xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_n^{\alpha_n}.$$

Let's assume that

1. $R_0(x, \xi)$ for all $\xi \in R^n/0$ is a bounded operator $H_0 \rightarrow H_{2m}$, and also

$$\sum_{i=0}^{2m} |\xi|^i \|R_0\|_{H_0 \rightarrow H_{2m-i}} \leq \delta_1$$

where $\delta_1 = \text{const}$, doesn't depend on x, ξ .

2. There is a ray $l = \{\lambda : \arg \lambda = \beta\}$ of complex plane λ such that the operator $R(x, \xi, \lambda) = \left[\sum_{|\alpha| \leq 2m} A_\alpha(i\xi)^\alpha - \lambda E \right]^{-1}$ is a bounded operator $H_0 \rightarrow H_{2m}$, for $\lambda \in l, \xi \in R^n, |x| > c$ and

$$(|\gamma(x)| + |\lambda|) \|R\|_{H_0 \rightarrow H_0} + \sum_{i < 2m} |\xi|^{2m-i} \|R\|_{H_0 \rightarrow H_{2m-i}} \leq \delta_2.$$

3. The values $\sup_{|x-x_0| \leq h} \|A_\alpha(x) - A_\alpha(x_0)\|$, $\sup_{|x-x_0| \leq h} \left| \frac{\gamma(x) - \gamma(x_0)}{\partial(x_0)} \right|$ approach zero for $h \rightarrow 0$ evenly by $x_0, |\alpha| = 2m$.

$$4. \quad \sum_{0 < |\alpha| \leq 2m} \|A_\alpha(x)\| \leq \delta_3.$$

Let's set H_i the space with the scalar product:

$$(f, g)_{H_i} = \int_{R^n} (f(x), g(x))_{H_i} dx, \quad f, g \in H_i, i = 0, 1, \dots, 2m$$

Main Results. The following theorems hold.

Theorem 1. *Let the conditions 1-4 fulfilled and*

$$\sum_{|\alpha| \leq 2m} \int_{R^n} \|D^\alpha u\|_{H_0}^2 dx + \int_{R^n} |\gamma^2(x)|^2 \|u\|_{H_0}^2 dx < \infty$$

for $\lambda \in l, |\lambda| \geq \lambda_0$.

Then

$$\sum_{|\alpha| \leq 2m} \int_{R^n} \|D^\alpha u\|_{H_0}^2 dx \leq c \|(L - \lambda E)u\|_{H_0}^2$$

where $c = \text{const}$, doesn't depend on λ .

Theorem 2. *If the conditions of Theorem 1 are fulfilled then the operator $(L - \lambda E)^{-1}$ is a bounded operator $H_0 \rightarrow H_0$ for every $\lambda \in l, |\lambda| \geq \lambda_0$.*

References

- [1] S. Agmon, *Lectures on elliptic value problem*, Van Nostrand Mathematical Studies, Princeton, 1965.
- [2] J.E. Allahverdiyev, On completeness of the system of eigen and adjoint vectors of one class non-selfadjoint operators depending on parameter, *DAN USSR*, 196(6), (1965), 1231-1234.
- [3] F. Browder, On the eigenfunctions and eigenvalues of the general linear elliptic differential operator, *Proc. Nat. Acad. Sci.*, 39, 433-439, (1953).
- [4] M.G. Gasymov, On the solvability of boundary-value problems for a class of operator differential equations, *DAN SSSR* 235(3), (1977), 505-508.
- [5] M.V. Keldysh, On the completeness of the eigenfunctions of some classes of non-selfadjoint operators *UMM*, 26(4), (1971), 15-41.

- [6] S.S. Mirzoev , On conditions of correct solvability of boundary value problems for operator-differential equations, *DAN SSSR* 273(2), (1983), 292-295.
- [7] A.A.Shkalikov , Elliptic equations in Hilbert space and related spectral problems, *Proc. of I.G.Petrovsky workshop*. Issue 14 140-224 (1989).

The Poisson Equation in a Rectangle with Nonlocal Boundary Conditions

Özlem Baksi^a, Kemal Demir^a, Baharchin Akhmadli^{b,c}

^a*Yildiz Technical University, Istanbul, Türkiye*

^b*Institute of Mathematics, Baku, Azerbaijan*

^c*Azerbaijan National Aviation Academy, Baku, Azerbaijan*

(e-mail: baksi@yildiz.edu.tr, kemal.demir@yildiz.std.edu.tr,
baharchin.ahmadli@naa.edu.az)

In this work, we investigate the strong solvability of a nonlocal boundary value problem for the Poisson equation in a rectangular domain. We utilize the Fourier spectral method together with the basis properties of the corresponding function system. We establish the uniqueness and existence of a strong solution in weighted grand Sobolev spaces.

References

- [1] T. Gasymov, B. Akhmadli, I. Feyzullayev, *On the Strong Solvability of a Nonlocal Boundary Value Problem for the Laplace Equation in Weighted Grand Lebesgue Spaces in Rectangle*, Journal of Contemporary Applied Mathematics, Vol. 9, No. 1, 2026.
- [2] B. T. Bilalov, S. R. Sadigova, *On solvability in the small of higher order elliptic equations in grand-Sobolev spaces*, Complex Variables and Elliptic Equations, 63(3), 2117–2130, 2021.

Solvability of a Nonlocal Problem in Orlicz-Sobolev Spaces

Bilal Bilalov^{a,b,c}, Ümit Ildiz^{a,d}, Yonca Sezer^a

^a*Yıldız Technical University, Istanbul, Türkiye*

^b*Institute of Mathematics, Baku, Azerbaijan*

^c*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*

^d*Marmara University, Istanbul, Türkiye*

(e-mail: ysezer@yildiz.edu.tr)

This study investigates a nonlocal problem for the Laplace equation in an unbounded domain. The concept of a strong solution is introduced, and the problem is analyzed by means of the spectral method. The analysis relies on the fact that the system of root functions associated with the corresponding spectral problem forms a basis in the Orlicz space. By employing the concept of Boyd indices for symmetric spaces, the strong solvability of the problem is established in Sobolev spaces generated by the Orlicz norm. Furthermore, the existence and uniqueness of strong solutions in Orlicz-Sobolev spaces are proved.

Keywords: Orlicz-Sobolev space; Laplace equation, nonlocal problem, strong solution

References

- [1] B.T. Bilalov, N.P. Nasibova, L.G. Softova, S. Tramontano, *Global solvability of the Laplace equation in weighted Sobolev spaces*, arXiv preprint arXiv:2307.07776, 2023.
- [2] B.T. Bilalov, S.R. Sadigova, Y. Sezer, N.P. Nasibova, *On solvability of polyharmonic Dirichlet problem in symmetric Sobolev spaces*, Mathematical Methods in the Applied Sciences, 47 (2024), 14386–14401.
- [3] B.T. Bilalov, Y. Sezer, F.A. Alizadeh, U. Ildiz, *Solvability of Riemann boundary value problems and applications to approximative properties of perturbed exponential system in Orlicz spaces*, Azerbaijan Journal of Mathematics, 14 (2024), 164–185.

- [4] B.T. Bilalov, Y. Sezer, U. Ildiz, T. Hagverdi, *On the basicity of one trigonometric system in Orlicz spaces*, Transactions Issue Mathematics, 44 (2024), 31–40.
- [5] A.V. Bitsadze, A.A. Samarskii, *On some simplest generalizations of linear elliptic boundary value problems*, Doklady AN SSSR, 185 (1969), 739–740.
- [6] F.I. Frankl, *Flow around airfoils by a stream of subsonic velocity with supersonic zones terminating in a straight-line condensation shock*, Prikladnaya Matematika i Mekhanika, 20 (1956), 196–202.
- [7] F.I. Frankl, *Selected Works on Gas Dynamics*, Nauka, Moscow, 1973.
- [8] N.I. Ionkin, E.I. Moiseev, *A problem for the heat transfer equation with two-point boundary conditions*, Differential Equations, 15 (1979), 1284–1295.
- [9] M.E. Lerner, O.A. Repin, *On Frankl-type problems for some elliptic equations with degeneration of various types*, Differential Equations, 35 (1999), 1087–1093.
- [10] E.M. Mamedov, N.P. Nasibova, Y. Sezer, *Some remarks on integral operators in Banach function spaces and representation theorems in Banach–Sobolev spaces*, Azerbaijan Journal of Mathematics, 14 (2024), 189–204.
- [11] E.I. Moiseev, *The solution of a nonlocal boundary value problem by the spectral method*, Differential Equations, 35 (1999), 1094–1100.

Solvability of a Class of a Linear Fourth-Order Boundary Value Problems in Hilbert Spaces

Mohammed Benharrat

National Polytechnic School of Oran-Maurice Audin, Oran, Algeria
(e-mail: mohammed.benharrat@enp-oran.dz)

The main purpose of this talk is to investigate the solutions of the following fourth-order abstract linear differential equation under various sets of

assumptions:

$$u^{(4)}(x) - 2Bu''(x) - Au(x) = f(x), \quad x \in (0, 1), \quad (1)$$

where $u(x)$ is a vector-valued function taking values in a suitable Hilbert space $\mathcal{H}$ (finite or infinite dimensional), A and B are linear (bounded or unbounded) accretive operators on $\mathcal{H}$, and the function f belongs to $L^p(0, 1; \mathcal{H})$ for some $1 < p < +\infty$.

When this problem is supplemented with appropriate nonhomogeneous boundary conditions, we establish existence, uniqueness, and maximal regularity of the solution under necessary and sufficient conditions on the data. We also provide an explicit representation formula using tools such as analytic semigroups, sectorial operators with bounded imaginary powers, the theory of strongly continuous cosine operator functions, and the perturbation theory of m -accretive operators.

Using the splitting method introduced by Krein [6], the equation is transformed into a system of two coupled second-order equations. The first equation is elliptic, while the second one is hyperbolic. These equations are not directly amenable to standard methods. The main difficulty lies in analyzing these two types of problems simultaneously.

In our analysis, the coupled second-order hyperbolic and elliptic equations are solved in order to obtain an explicit solution to problem (1) under suitable boundary conditions. Our approach is primarily based on fractional powers of m -accretive operators, their perturbation theory, the techniques involving holomorphic semigroups generated by such operators, and the theory of strongly continuous cosine operator functions.

References

- [1] M. Benharrat, F. D. M. Bezerra, *Fractional powers counterpart of third order abstract Cauchy problems and of maximal accretive operators*, Quaestiones Mathematicae, 1–20, 2026.
- [2] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Birkhäuser, Basel, 2001.

- [3] F. Bouchelaghem, M. Benharrat, *The Moore-Penrose inverse and accretive operators with application to quadratic operator pencils*. Filomat 36 (7), pp. 2475–2491, 2022.
- [4] F. Bouchelaghem, M. Benharrat, *A factorization of a quadratic pencils of accretive operators and applications*. Mediterr. J. Math. 19, 9 (2022).
- [5] M. Benharrat, F. Bouchelaghem, A. Thorel, *On the solvability of fourth-order boundary value problems with accretive operators*, Semigroup Forum, 2023.
- [6] S. G. Krein, *Linear Differential Equations in Banach Spaces*, Moscou, 1967.

Semi Cauchy Variants of Alpha Convergence and Exhaustiveness with Korovkin-Type Theorems

Alper Erdem

Mersin University, Mersin, Türkiye
(e-mail: alpererdem@mersin.edu.tr)

This study considers semi Cauchy versions of alpha convergence and the notion of exhaustiveness for sequences of functions defined between metric spaces and examines their connections with existing notions. Moreover, Korovkin-type approximation theorems in the sense of some notions of exhaustiveness, is established, showing that the classical conditions in the Korovkin theorem can be extended.

References

- [1] A. Erdem, A. Arpacioğlu, T. Tunç, *On Semi Uniform Alpha Convergence and a Korovkin-Type Theorem*, Dolomites Research Notes on Approximation, 2026, 19(2), 44–51. DOI: 10.25430/pupj-DRNA-2026-2-7.
- [2] A. Erdem and T. Tunç, *On Various Modes of Convergence and Notions of Exhaustiveness With Korovkin-Type Theorems*, Journal of Mathematics, 2026(1), 8881306.

- [3] V. Gregoriades and N. Papanastassiou, *The notion of exhaustiveness and Ascoli-type theorems*, Topology and its Applications, 2008, 155(10), 1111–1128.
- [4] N. Papanastassiou, *A note on convergence of sequences of functions*, Topology and its Applications, 2020, 275, 107017.

A Note On Convergence Notions Invariant Under Uniform Conjugacy

Alper Erdem^a, Engin Furat^b

^a*Mersin University, Mersin, Türkiye*

^b*Eskişehir Technical University, Eskişehir, Türkiye*

(e-mail: alpererdem@mersin.edu.tr, efuratt@gmail.com)

In the literature, the notions of uniform conjugate and topological conjugacy have been investigated, and the behaviors of sequence of maps that are chaotic in the sense of Devaney under these notions have been researched. In this study, the properties of uniform alpha convergence, Cauchy alpha convergence, and uniform exhaustiveness of sequences of functions under uniform conjugate and topological conjugate transformations are examined. Additionally, counterexamples showing that these properties are not preserved under topological conjugate transformations are presented.

References

- [1] R. Devaney, *An introduction to chaotic dynamical systems*, CRC press, 2018.
- [2] A. Erdem and T. Tunç, *On Cauchy alpha and uniform alpha convergence*, Filomat, 2025, 39(20), 7033–7042.
- [3] V. Gregoriades and N. Papanastassiou, *The notion of exhaustiveness and Ascoli-type theorems*, Topology and its Applications, 2008, 155(10), 1111–1128.

- [4] C. Tian and G. Chen, *Chaos of sequence of maps in a metric space*, Chaos, Solitons and Fractals, 2006, 28, 1067–1075.

On Asymptotics of the Green Function Cauchy Problem for Parabolic Equation Corresponding Higher Order Operator-Differential Equation in Hilbert Space

Nigar Gadirli

Sumgait State University, Sumgait, Azerbaijan
(e-mail: nigarqadirli@sdu.edu.az)

Let H be a separable Hilbert space. Consider the operator L in the space $H_1 = L_2(H; R^n)$, defined by differential expression.

$$l(y) = (-1)^m \sum_{k_1+k_2+\dots+k_n=2m} A^{k_1,k_2,\dots,k_n}(x) \frac{\partial^{2m} u}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} + Q(x)u. \quad (1)$$

Here, $x = (x_1, \dots, x_n) \in R^n$ and $A^{k_1,k_2,\dots,k_n}(x)$ are real-valued functions defined on the entire space R_n and satisfying the Lipschitz condition. Moreover, condition

$$C_1 |S|^{2m} \leq \sum_{k_1+k_2+\dots+k_n=2m} A^{k_1,k_2,\dots,k_n}(x) S_1^{k_1} \cdot S_2^{k_2} \dots S_n^{k_n} \leq C_2 |S|^{2m}.$$

is assumed to hold.

In the present study, we consider the Cauchy problem

$$\frac{\partial u}{\partial t} = -Lu = -(L_0 + Q(x))u, \quad (2)$$

$$u(x, 0) = \psi(x), \quad \psi(x) \in L_2(R_n; H). \quad (3)$$

corresponding to the differential expression (1).

The Green's function of the Cauchy problem (2), (3) is sought in the form

$$G(x, y, t) = G_1(x - y, y, t)e^{-tQ(y)} + \int_0^t d\tau \int_{R^n} G_1(x - \xi, \xi, t - \tau)e^{-(t-\tau)Q(\xi)} \varphi(\xi, y, \tau) d\xi. \quad (4)$$

by means of the parametrix method [2]. Here $G_1(x - y, y, t) = R(x - y)G_0(x - y, y, t)$.

Furthermore

$$R(x) = \begin{cases} 1, & |x| \leq \frac{1}{2}, \\ 0, & |x| > 1. \end{cases}$$

Where $G_0(x - y, y, t)$ denotes the Green's function of the problem

$$\begin{cases} \frac{\partial u}{\partial t} = -L_0\left(\eta, \frac{\partial}{\partial x}\right), \\ u(x, 0) = \psi(x), \quad \psi(x) \in L_2(R_n; H). \end{cases}$$

It is shown that the following asymptotic formula holds for the Green's function:

$$G(x, y, t) = G_0(x - y, y, t)e^{-tQ(y)} + t^{1-b} \exp\left[-c \frac{|x - y|^r}{t^{1/(2m-1)}}\right] \cdot \|e^{-ctQ(y)}\|_1 + \|Q^{-l}(y)\| \cdot O(1). \quad (5)$$

Where

$$G_0(x - y, y, t) = \frac{1}{(2\pi)^n} \int_{R_n} \exp[tL_0(y, s) + i(s, x - y)] ds.$$

By making use of this equality, it is proved that the spectrum of the operator L is discrete [1].

References

- [1] G.I. Aslanov, Sh.G. Baimov, *Distribution of eigenvalues of some non-self-adjoint operators*, Izvestiya AN Azerb. SSR, Series of Physical-Technical and Mathematical Sciences, 1974, No. 6, pp. 59–64.

- [2] A.G. Kostyuchenko, *Asymptotic behavior of the spectral function of self-adjoint elliptic operators*, In: “Fourth Mathematical School”, Kiev, 1968, pp. 42–117.
- [3] N.A. Gadirli, *Distribution of eigenvalues of high-order elliptic operator-differential equations in Hilbert space*, Journal of Qafqaz University, Mathematics and Computer Science, Baku, 2016, No. 2, pp. 177–184.
- [4] H.I. Aslanov, N.A. Gadirli, *On asymptotics of the function of distribution of spectrum for higher order partial operator-differential equations in Hilbert space*, IX International Conference of the Georgian Mathematical Union, Batumi–Tbilisi, 2018, p. 77.

On the Strong Solvability of a One-Dimensional Parabolic Problem With a Non-Classical Boundary Condition

Telman Gasymov^{a,b} Ilkin Feyzullayev^b, Amiragha Shikhammadov^c

^a*Baku State University, Baku, Azerbaijan*

^b*Institute of Mathematics, Baku, Azerbaijan*

^c*Guba branch of Azerbaijan State Pedagogical University, Guba, Azerbaijan*

(e-mail: telmankasumov@rambler.ru, ilkin0704@mail.ru,
amiragashixemmedov@mail.ru)

Introduction. Solving some partial differential equations with nonlocal boundary conditions using the Fourier method leads to spectral problems with boundary conditions that are regular, but not strongly regular. For this reason, the root functions of these problems typically do not form a basis in the corresponding function space. In such cases, direct application of the Fourier method is impossible. Based on these eigenfunctions, a new system of functions is constructed, consisting of linear combinations of the root functions, which already forms a basis in $L_p[0, 1]$. However, the resulting system is not a system of eigenfunctions of the spectral problem. Nevertheless, this system is used to solve the equation under consideration using the Fourier method. One

such problem is the following initial-boundary value problem for a parabolic equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2}, \quad 0 < x < 1, \quad 0 < t < T, \quad (1)$$

with the initial condition

$$u(x, 0) = f(x), \quad 0 \leq x \leq 1, \quad (2)$$

and the boundary conditions

$$u(0, t) = u(1, t), \quad \frac{\partial u}{\partial x}(0, t) = \beta \frac{\partial u}{\partial x}(1, t) + au(1, t), \quad 0 \leq t \leq T. \quad (3)$$

We study the question of the correctness of the boundary value problem, that is, the existence of a solution and its uniqueness in the Sobolev class $W_p^{2,1}(D)$, $1 < p < +\infty$, where $D = (0, 1) \times (0, T)$, and stability with respect to the initial data. Recall that a strong solution of problem (1)-(3) is a function $u(x, t) \in W_p^{2,1}(D)$ that satisfies equation (1) almost everywhere in D and conditions (2) and (3) in the sense of traces.

In the classical formulation, similar problems in the case of other boundary conditions are considered in works [1-3]. The peculiarity of problem (1)-(3) is that for $a, \beta \in C, \beta \neq \pm 1$, the boundary conditions (3) are regular in the sense of Birkhoff, but are not strongly regular for any value of $a \neq 0$, and the eigenfunctions of the corresponding spectral problem do not form a basis in any space $L_p[0, 1]$, $1 < p < \infty$.

Main results. Using the Fourier method, the question of the solvability of problem (1)-(3) is reduced to the following spectral problem

$$Lu(x) = -u''(x) = \lambda u(x), \quad 0 < x < 1, \quad u(0) = u(1), \quad u'(0) = \beta u'(1) + au(1). \quad (4)$$

The spectral properties of this problem depend significantly on the values of the coefficients $a, \beta \in C$. For $\beta = -1$, the boundary conditions are not regular. For any $a, \beta \in C, a \neq 0, \beta \neq \pm 1$, all eigenvalues of this operator can be distributed into two series, which have the form

$$\lambda_{k,1} = (\rho_{k,1})^2, \quad k \in N, \quad \lambda_{k,2} = (\rho_{k,2})^2, \quad k \in Z^+, \quad (5)$$

where $\rho_{k,1} = 2pk$ and $\rho_{k,2}$ are the roots of the equation

$$\operatorname{tg} \frac{\rho}{2} = \frac{\alpha}{(1 + \beta)\rho}, \quad \rho > 0. \quad (6)$$

Each eigenvalue of the operator L corresponds to a unique (up to a non-zero factor) eigenfunction. Using the numbering introduced by equalities (5), in the case $\beta \neq \pm 1$, the set of eigenfunctions can be represented as

$$u_{k,1}(x) = \cos 2\pi kx + \frac{a}{(1-\beta)\pi k} \sin 2\pi kx, \quad k \in N;$$

$$u_{k,2}(x) = \cos 2\pi kx + \frac{a}{2(1+\beta)\pi k} \sin 2\pi kx + O\left(\frac{1}{k^2}\right) \quad k \in Z^+. \quad (7)$$

When constructing a solution to problem (1),(2) using the separation of variables method, the basis property of the system of eigenfunctions of the operator L is of fundamental importance.

Theorem 1. *For any $a, \beta \in C$, $a \neq 0$, $\beta \neq \pm 1$, the system of functions (7) is almost normalized, complete, and minimal, but does not form a basis in the space $L_p[0, 1]$, $1 < p < +\infty$.*

Theorem 1 means that it is impossible to use system (7) to solve problem (1)-(3) by separation of variables. To eliminate this drawback, we introduce a new system $\{w_n(x)\}_{n \in Z_+}$ as follows:

$$w_0(x) = u_{0,2}(x), w_{2k}(x) = u_{k,1}(x), w_{2k-1}(x) = (u_{k,2}(x) - u_{k,1}(x))\beta_k, k \in N, \quad (8)$$

where indicated $\beta_k = \frac{p(\beta^2-1)}{a\beta}k$.

Theorem 2. *Let $a \neq 0$, $\beta \neq \pm 1$. Then the system $\{w_n(x)\}_{n \in Z_+}$, defined by equalities (8), forms a basis in the space $L_p(0, 1)$, $1 < p < +\infty$, equivalent to the classical trigonometric system.*

The following theorem has been proven regarding the solvability of problem (1)-(3).

Theorem 3. *Let $\varphi(x) \in W_p^{2+\gamma}(0, 1)$, where $\gamma = 0$ for $1 < p \leq 2$; $0 < \gamma < 1$ for $2 < p \leq 3$ and $1 - \frac{3}{p} < \gamma < 1$ for $p > 3$; in addition, the function $\varphi(x)$ satisfies the boundary conditions (4), and for $\frac{1}{p} < \gamma < 1$ it additionally satisfies the condition $\varphi''(0) = \varphi''(1)$. Then problem (1)-(3) has a unique strong solution from the class $W_p^{2,1}(D)$ and for this solution the following estimate holds:*

$$\|u\|_{W_p^{2,1}(D)} \leq C \|\varphi\|_{W_p^{2+\gamma}(0,1)},$$

where the constant C does not depend on the functions $\varphi(x)$ and $u(x, t)$.

References

- [1] Ionkin N.I. Solution of a boundary value problem of heat conduction theory with a nonclassical boundary condition. *Differential Equations*, 1977, 13(2), 294-304.
- [2] Ionkin N.I., Moiseev E.I. On a problem for the heat equation with two-point boundary conditions. *Differential Equations*, 1979, 15(7), 1284-1295.
- [3] Mokin A.Y. Correctness of task family with nonclassical boundary conditions. *Comput. Res. Model.* 2009, 1(2), 139-146 (in Russian)

On Convergence of the Spectral Expansion in Root Vector Functions of Discontinuous Schrödinger Operator With Matrix Potential

Elchin Ibadov

Azerbaijan State Pedagogical University, Baku, Azerbaijan
(e-mail: e.c.ibadov@yahoo.com)

Let $G = \bigcup_{l=1}^m G_l, G_l = (\xi_{l-1}, \xi_l), a = \xi_0 < \xi_1 < \dots < \xi_m = b$, where $-\infty < a < b < +\infty$.

By $W_{p,n}^{1,m}(G) = W_{p,n}^1(G, \{\xi_l\}_{l=0}^m)$, $p \geq 1$ we denote a class of n -component vector functions satisfying the following property:

$f(x) = (f_1(x), f_2(x), \dots, f_n(x))^T \in W_{p,n}^{1,m}(G)$, then for every $l, l = \overline{1, m}$ there exists a vector function

$f_l(x) = (f_{l1}(x), f_{l2}(x), \dots, f_{ln}(x))^T, f_{li}(x) \in W_p^1(G_l), i = \overline{1, n}$ such that $f(x) = f_l(x)$ at $\xi_{l-1} < x < \xi_l$; $f(\xi_l) = f(\xi_l + 0), l = \overline{0, m-1}, f(\xi_m) = f(\xi_m - 0)$. In addition, $W_p^1(G_l)$ is an ordinary Sobolev's class, and $W_{p,n}^{1,m}(G)$ is Sobolev type class of n -component vector functions defined on $G = [a, b]$. The norm of the element $f \in W_{p,n}^{1,m}(G)$ is defined by equality

$$\|f\|_{W_{p,n}^{1,m}(G)} = \sum_{l=1}^m \|f_l\|_{W_{p,n}^1(G_l)} = \sum_{l=1}^m \left(\|f_l\|_{p,n} + \|f'_l\|_{p,n} \right),$$

where

$$\|f_l\|_{p,n} = \left(\int_{G_l} \left(\sum_{i=1}^n |f_{li}(x)|^2 \right)^{p/2} dx \right)^{\frac{1}{p}} \quad \|f'_l\|_{p,n} = \left(\int_{G_l} \left(\sum_{i=1}^n |f'_{li}(x)|^2 \right)^{p/2} dx \right)^{\frac{1}{p}}.$$

Consider the discontinuous Schrödinger operator

$$L\psi = \psi'' + Q(x)\psi, \quad (1)$$

on one arbitrary finite interval $G = (a, b)$, where $\psi(x) = (\psi_1(x), \psi_2(x), \dots, \psi_n(x))^T$, $Q(x) = (q_{ij}(x))$, $i = \overline{1, n}$, $j = \overline{1, n}$ moreover $q_{ij}(x)$, are complex-valued summable functions on the interval G .

Following V.A. Il'in [1], a system of root vector functions of operator (1) is an arbitrary system $\{\psi_k\}_{k=1}^\infty$ of complex-valued identically nonzero vector functions from $W_{1,n}^{2,m}(G)$ that satisfy almost everywhere in G the equation

$$L\psi_k + \lambda_k\psi_k = \theta_k\psi_{k-1},$$

where θ_k is either 0 (in which case ψ_k is called an eigen vector function) or 1 (in which case require that $\lambda_k = \lambda_{k-1}$ and ψ_k is called an associated vector function of order r , where $\theta_k = \theta_{k-1} = \dots = \theta_{k-r+1}$ and $\theta_{k-r} = 0$).

Let denote $\mu_k = \sqrt{\lambda_k}$, where $\mu_k = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let the system $\{\varphi_k(x)\}_{k=1}^\infty$ consist of root vector functions formally adjoint to the operator (1) $L^*\varphi = \varphi'' + Q^*(x)\varphi$, where $Q^*(x) = \overline{Q(x)}$ is matrix-function adjoint to $Q(x)$, satisfying equation $L^*\varphi_k + \overline{\lambda_k}\varphi_k = \theta_{k+1}\varphi_{k+1}$ almost everywhere in G .

For any vector function $f(x) \in W_{p,n}^{1,m}(G)$, $p \geq 1$, compose partial sum of its biorthogonal expansion by system $\{\psi_k(x)\}_{k=1}^\infty$,

$$\sigma_\nu(x, f) = \sum_{\operatorname{Re} \mu_k \leq \nu} (f, \varphi_k) \psi_k(x), \quad \nu \geq 1,$$

where

$$(f, \varphi_k) = \sum_{l=1}^m \int_{G_l} \langle f(x), \varphi_k(x) \rangle dx, \quad \langle f(x), \varphi_k(x) \rangle = \sum_{i=1}^n f_i(x) \overline{\varphi_{ki}(x)},$$

$$\psi_k(x) = (\psi_{k1}(x), \psi_{k2}(x), \dots, \psi_{kn}(x))^T.$$

Suppose

$$R_\nu(x, f) = f(x) - \sigma_\nu(x, f)$$

and set

$$Q_m(f, \varphi_k) = \langle \varphi'_k(b-0), f(b) \rangle - \langle \varphi'_k(a+0), f(a) \rangle + \\ + \sum_{l=1}^{m-1} \left[\langle \varphi'_k(\xi_l-0), f(\xi_l-0) \rangle - \langle \varphi'_k(\xi_l+0), f(\xi_l+0) \rangle \right].$$

Theorem. Let the system $\{\psi_k(x)\}_{k=1}^\infty$ of root vector functions of operator (1) satisfy the following conditions:

1. The system $\{\psi_k(x)\}_{k=1}^\infty$ is complete and minimal in $L_2^n(G)$;
2. The inequalities

$$|Im\mu_k| \leq const, \quad (2)$$

$$\sum_{\tau \leq Re\mu_k \leq \tau+1} 1 \leq const, \quad \forall \tau \geq 0, \quad (3)$$

3. There exist a constant C_0 , such that

$$\|\psi_k\|_{2,n} \|\varphi_k\|_{2,n} \leq C_0, \quad k = \overline{1, \infty}, \quad (4)$$

4. For the vector function $f(x) \in W_{p,n}^{1,m}(G)$, $p > 1$, there exists a constant $C_1(f)$, such that the inequality

$$|Q_m(f, \varphi_k)| \leq C_1(f) \|\varphi_k\|_{2,n}, \quad k = \overline{1, \infty}, \quad (5)$$

holds.

Then biorthogonal expansion of the vector function $f(x)$ by the system $\{\psi_k(x)\}_{k=1}^\infty$ converges absolutely and uniformly on $\overline{G} = [a, b]$ and the following relations hold:

$$f(x) = \sum_{k=1}^{\infty} (f, \varphi_k) \psi_k(x), \quad x \in \overline{G}, \quad (6)$$

$$\sup_{x \in \overline{G}} |R_\nu(x, f)| \leq const \left\{ \nu^{-\delta} \|f\|_{W_{p,n}^{1,m}(G)} + \nu^{-1} \left[C_1(f) + \|Q\|_1 \|f\|_{\infty,n} \right] \right\}, \quad (7)$$

$$\sup_{x \in \overline{G}} |R_\nu(x, f)| = o(\nu^{-\delta}), \quad \nu \rightarrow +\infty, \quad (8)$$

where $\delta = \min \left\{ \frac{1}{2}, \frac{1}{q} \right\}$, $p^{-1} + q^{-1} = 1$, the symbol "0" depends on expanded vector function $f(x)$, const doesn't depend on expanded vector function $f(x)$, $\|Q\|_1 = \int_a^b \sum_{i,j=1}^n |q_{ij}(x)| dx$.

References

- [1] V. A. Il'in, Necessary and sufficient conditions for being a Riesz basis of root vectors of second-order discontinuous operators, *Differentsial'nye Uravneniya*, 1986, 22(12), 2059–2071

Grand Classes of Hardy and the Cauchy Formula

Migdad Ismailov^a, Chinara Hasanova^b,

^a*Azerbaijan State Oil and Industry University, Baku, Azerbaijan*

^b*Ganja State University, Ganja, Azerbaijan*

(e-mail: migdad-ismailov@rambler.ru, Chinareh@gmail.com)

Let $1 < p, v < +\infty$. We denote by $L_p(-\pi, \pi)$ the grand Lebesgue space ([1]) of measurable on $[-\pi, \pi]$ functions f with finite norm

$$\|f\|_{L_p} = \sup_{0 < \varepsilon < p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}},$$

and l_v is the Banach space ([2]) of all sequences of scalars $\{a_n\}_{n \in \mathbb{Z}}$ with finite norm

$$\|\{a_n\}_{n \in \mathbb{Z}}\|_{l_v} = \sup_{0 < \varepsilon < p-1} \varepsilon^{\frac{1}{v(\varepsilon)}} \left(\sum_{n=-\infty}^{+\infty} |a_n|^{v'(\varepsilon)} \right)^{\frac{1}{v'(\varepsilon)}},$$

here $\frac{1}{v} + \frac{1}{v'} = 1$. Consider the Banach space $L_{p,v}(-\pi, \pi)$ ([2]) of functions $f \in L_p(-\pi, \pi)$ with finite norm

$$\|f\|_{L_{p,v}} = \|f\|_{L_p} + \|\hat{f}\|_{l_v},$$

where $\hat{f} = \{f_n\}_{n \in \mathbb{Z}}$ and f_n are the Fourier coefficients of f according to system $\{e^{int}\}_{n \in \mathbb{Z}}$.

It is known ([4]) that for a function f analytic in a disk $\omega = \{z \in \mathbb{C} : |z| < 1\}$ to belong to the Hardy class H^p , it is necessary and sufficient that its almost everywhere on $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ non-tangential boundary values $f^+(e^{it}) \in L_p(-\pi, \pi)$, and that the Cauchy formula holds:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f^+(\xi)}{\xi - z} d\xi, z \in \omega. \quad (1)$$

In this paper, this result is established in the space $L_{p, \nu}(-\pi, \pi)$.

Definition 1. We say that a function f is analytic in the disk belongs to the grand Hardy class $H_{p, \nu}^+$ if the condition

$$\|f\|_{H_{p, \nu}^+} = \sup_{0 < r < 1} \|f_r\|_{p, \nu} < +\infty$$

is satisfied, where $f_r(t) = f(re^{it})$.

The following theorem proves that the class $H_{p, \nu}^+$ is a subclass of the grand Hardy class H_p^+ ([3]).

Theorem 1. Let $f \in H_{p, \nu}^+$ and $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $z \in \omega$. Then $f \in H_p^+$ and the equality $\|f\|_{H_{p, \nu}^+} = \|f\|_{H_p^+} + \left\| \{a_n\}_{n \in \mathbb{Z}_+} \right\|_{l_{\nu}}$ holds.

The following theorem is true.

Theorem 2. The following statements are valid.

1) if $f \in H_{p, \nu}^+$, then $f^+(e^{it}) \in L_{p, \nu}(-\pi, \pi)$ and the Cauchy formula (1) holds;

2) if $f^+(e^{it}) \in L_{p, \nu}(-\pi, \pi)$, then the function f defined by formula (1) belongs to class $H_{p, \nu}^+$ and its a.e. on γ non-tangential boundary values are $f^+(e^{it})$.

Note that these results in ordinary Hardy classes with summable Fourier coefficients were studied in [5].

This work was supported by the Science Foundation of Azerbaijan, Grant No. AEF-MGC-2025-1(54)-20/03/1-M-03.

References

- [1] R.E. Castilo and H. Rafeiro, An Introductory Course in Lebesgue Spaces, Springer Int. Publ. Switzerland, 2016.
- [2] M.I. Ismailov, C.M. Hasanova, H. Z. Ashirova, On some properties of grand sequence spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, 45 (4) (2025), 88-96.
- [3] M.I. Ismailov, On the Solvability of Riemann Problems in Grand Hardy Classes, Mathematical Notes, 108 (4) (2020), 523-537
- [4] P. Kusiś. Introduction to the theory of spaces H^p . M., "Mir", 1984.
- [5] Bilalov B.T. Some questions of approximation. Baku: Elm, 2016.

Some Properties of X -valued Poisson Integrals in Bochner Spaces

Zaur Kasumov^a, Nigar Ahmedzade^a, Tarlan Garayev^b

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Karabakh University, Khankendi, Azerbaijan*

(e-mail: zaur@celt.az, nigar_sadigova11@mail.ru,
tarlan.garayev@karabakh.edu.az)

It is considered X -valued Poisson integrals related to the boundary value problem for elliptic equation with constant coefficients in n -dimensional half plane R_+^n . We establish bounded properties of these operators concerning Bochner space.

In this work we will use the following notations. R —real numbers; $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ —open ball in R^n , where $|x| = \sqrt{x_1^2 + \dots + x_n^2}$, $x = (x_1, \dots, x_n) \in R^n$; $R_+^n = \{x = (x_1; \dots; x_n) \in R^n : x_n > 0\}$ — n -dimensional half space; $\Omega_r(x_0) = \Omega \cap B_r(x_0)$; $B_r = B_r(0)$; B -space—Banach space; $\|\cdot\|_X$ —norm in X .

Let's recall the definition of UMD property.

Definition 1. A B -space X is said to have the property of UMD, if for all $p \in (1, \infty)$ there exists a finite constant $\beta \geq 0$ (depending on p and X) such that the following holds. Whenever $(S; A; \mu)$ is a σ -finite measure space, $\{F_n\}_{n=0}^N$ is a σ -finite filtration and $\{f_n\}_{n=0}^N$ is a finite martingale in $L_p(S; X)$, then for all scalars $|\varepsilon_n| = 1$; $n = \overline{1, N}$, we have

$$\left\| \sum_{n=1}^N \varepsilon_n df_n \right\|_{L_p(S; X)} \leq \beta \left\| \sum_{n=1}^N df_n \right\|_{L_p(S; X)},$$

where $df_n = f_n - f_{n-1}$ is a martingale difference.

Consider the integral

$$u(x; t) = \int_{R^{n-1}} K(x - y; t) f(y) dy,$$

where the kernel $K(x; t)$ satisfies the following conditions.

(i) K is homogeneous with degree $(-n)$ and has a form

$$K(x; t) = \frac{\Omega\left(\frac{x}{|P|}; \frac{t}{|P|}\right)}{(|x^2 + t^2|)^{\frac{n-1}{2}}},$$

where $x = (x_1; \dots; x_{n-1}) \in R^{n-1} : |x|^2 = \sum_1^{n-1} |x_i|^2$; $P = (x; t) \in R_+^n : |P|^2 = |x|^2 + t^2$; $\sigma_r^+ = \{P \in R_+^n : |P| = r\}$; $B'_r = \{x \in R^{n-1} : |x| < r\}$; $P' = (x; 0) \in R^{n-1}$.

(ii) $\Omega(\cdot; \cdot)$ is continuous on σ_1^+ and there exists constants $\wp > 0$ & $\alpha' \in (0, 1]$, such that

$$|\Omega(P') - \Omega(Q)| \leq \wp \overline{P'Q}^{\alpha'}; \max_{\sigma_1} |\Omega| \leq \wp;$$

for $\forall P' = (x; 0) \in \sigma_1^+$ & $Q = (y; t) \in \sigma_1^+$, where $\overline{P'Q}$ denotes the geodetic distance between points P' and Q on σ_1^+ ;

(iii)

$$\int_{|x|=1} \Omega(x; 0) d\sigma_x = 0,$$

where $d\sigma_x$ is an area element of sphere $|x| = 1$.

The following theorem is proved.

Theorem 1. *Let X be UMD space with separable dual X^* and*

$$K(x; t) = \frac{\Omega\left(\frac{x}{|x|}\right)}{(|x|^2 + t^2)^{\frac{n-1}{2}}},$$

where the function Ω satisfies the conditions (i) – (iii).

Consider X -valued Poisson type integral

$$u(x; t) = \int_{R^{n-1}} K(x - y; t) f(y) dy.$$

Then there exists a constant $c > 0$ such that $(1 < p < \infty)$

$$\|u(\cdot; t)\|_{L_p(R^{n-1}; X)} \leq c \|f\|_{L_p(R^{n-1}; X)}, \forall t > 0; \forall f \in L_p(R^{n-1}; X).$$

This work was supported by the Azerbaijan Science Foundation-Grant No. AEF-MGC-2024-2(50)-16/02/1-M-02

References

- [1] T.J. Mammadov, *Strong solvability of a nonlocal problem for the Laplace equation in weighted grand Sobolev spaces*, Azerbaijan Journal of Mathematics, 2023, 13(1), 188-204.
- [2] B.T. Bilalov, T.M. Ahmadov, Y. Zeren, S.R. Sadigova, *Solution in the small and interior Schauder-type estimate for the m -th order elliptic operator in Morrey-Sobolev spaces*, Azerbaijan Journal of Mathematics, 2022, 12(2), 190–219.

Uniqueness and Existence of One Nonlocal Problem in Orlicz-Sobolev Space

Seda Kızılbudak Çalışkan^a, Sümeyye Gica^a,
Baharchin Akhmadli^{b,c}

^aYildiz Technical University, Istanbul, Türkiye

^bInstitute of Mathematics, Baku, Azerbaijan

^cAzerbaijan National Aviation Academy, Baku, Azerbaijan

(e-mail: skizilb@yildiz.edu.tr, ssumeyyetosuncu@yildiz.std.edu.tr,
baharchin.ahmadli@naa.edu.az)

This study examines a boundary problem with nonlocal boundary conditions for the Laplace equation in the semi-infinite domain $([0, 2\pi] \times [0, \infty))$. Utilizing the basis property of the root function system associated with the spectral problem and applying the spectral method, the existence and uniqueness of strong solutions in Orlicz-Sobolev spaces are established.

References

- [1] B Bilalov, S Sadigova, T Gasymov, I Feyzullayev, *On strong solvability of one boundary value problem for X -valued Laplace equation on half disc*, Integral Transforms and Special Functions, 2026, 1-26, DOI: 10.1080/10652469.2026.2694714
- [2] B Bilalov, Y Sezer, Ü Ildız *Strong Solvability of One Nonlocal Problem by the Spectral Method in Orlicz-Sobolev Spaces* Journal of Contemporary Applied Mathematics 16 (2), 2026, 13-33

Degree of Approximation for Harmonically Modulated Functions in the Hölder Class

Ahmet Köprücü, Uğur Değer

Department of Mathematics, Mersin University, Mersin, Türkiye
(e-mail: koprucuahmet@hotmail.com, udeger@mersin.edu.tr)

Classical Fourier approximation is based on partial sums and the Dirichlet kernel. Using this approach, Kuz'michev (2025) proved that functions and their derivatives in the Hölder class can be approximated by Fourier coefficients under harmonic modulation, and showed through examples that this method outperforms the classical Fourier series approximation [1].

In this study, inspired by the work of Kuz'michev, the approximation properties of functions in the Hölder class are investigated using the Fejer mean under harmonic modulation. By comparing the obtained results with previous findings, the efficacy of the approximation is demonstrated. Furthermore, the rate of convergence achieved via the Fejer mean under harmonic modulation for functions in the Hölder class is established in the theorem below.

Theorem 1. Let $f(y) \in C^\alpha(G)$. For any $\varepsilon > 0$ the function f can be approximately expressed by the following finite sum of the coefficients of the Fourier series of the harmonically modulated argument $y = x + h \cos t$:

$$\left| \frac{A_0(x, h)}{2} + \sum_{k=1}^N (-1)^k \left(1 - \frac{k-1}{N} \right) A_{2k}(x, h) - f(x) \right| = O(N^{-\alpha})$$

Here h is the modulation amplitude, $t \in [-\pi, \pi]$, $x \in [a, b]$, $y \in [a-h, b+h] = G$ and $A_m(x, h) = \frac{2}{\pi} \int_0^\pi f(x + h \cos t) \cos(mt) dt$.

References

- [1] N. D. Kuz'michev, Approximation of function and its derivative from the Hölder-Lipschitz class by their Fourier coefficients for the harmonically modulated argument, *Computational Mathematics and Mathematical Physics*, 65(4), 663–673, 2025.

Estimating the Local Equiconvergence Rate for the one – Dimensional Schrödinger Operator With Summable Potential

Veli Kurbanov^{a,b}, Leyla Buksayeva^b

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan State Pedagogical University, Baku, Azerbaijan*

(e-mail: q.vali@yahoo.com, leylabuksayeva.80@mail.ru)

Consider the one - dimensional Schrödinger operator

$$Lu = -u'' + q(x)u$$

with a complex - valued potential $q(x) \in L_r(G)$, $r \in [1, \infty]$, on the interval $G = (0, 1)$. Root functions (eigen and associated functions) of the operator L are understood in a generalized sense, i.e. irrespective of boundary conditions conditions (see [1,2]).

We will deal with the equiconvergence rate in metrics L_s , $s \in [1, \infty]$, of spectral expansion of the function $f(x) \in L_p(G)$, $p \geq 1$, in root functions of the operator L with trigonometric Fourier series on the compact $K \subset G$.

Let $\{u_k(x)\}_{k=1}^\infty$ be an arbitrary system consisting of eigen - and associated functions (root functions) of the operator L and $\{\lambda_k\}_{k=1}^\infty$ be the corresponding system of eigenvalues. We require that, together with each associated function of order m , $m \geq 1$, the system $\{u_k(x)\}_{k=1}^\infty$ also contains the corresponding eigenfunctions and all associated functions of order less than s . This means that each element of the system $\{u_k(x)\}_{k=1}^\infty$ is identically non-zero, is absolutely continuous together with its first order derivative on $\overline{G}$, and satisfies the equation $Lu_k = \lambda_k u_k + \theta_k u_{k-1}$ almost everywhere in G , where λ_k equals either 0 (in this case $u_k(x)$ is an eigenfunction) or 1 (in this case $u_k(x)$ is an associated function of order $l \geq 1$, and $u_{k-1}(x)$ is an associated function of order $l - 1$, $\lambda_k = \lambda_{k-1} = \dots = \lambda_{k-l}$, $\theta_k = \theta_{k-1} = \dots = \theta_{k-l+1} = 1$ with $\theta_{k-1} = 0$).

Let the system $\{u_k(x)\}_{k=1}^\infty$ and the system of eigenvalues $\{\lambda_k\}_{k=1}^\infty$ of the operator L satisfy the conditions A_p (II' in conditions):

- 1) $\{u_k(x)\}_{k=1}^\infty$ is closed and minimal in $L_p(G)$ for some fixed p , $p \geq 1$
- 2) Carleman's condition and the "sum of units" condition are satisfied

$$|Im \mu_k| \leq const, \quad k = 1, 2, \dots;$$

$$\sum_{\tau \leq \rho_k \leq \tau+1} 1 \leq \text{const}, \quad \forall \tau \geq 0$$

where $\mu_k^2 = \lambda_k$, $\rho_k = \operatorname{Re} \mu_k \geq 0$;

3) for every compact $K \subset G$ there exists a constant $C_0(K)$ such that

$$\|u_k\|_{p,K} \|v_k\|_q \leq C_0(K), \quad k = 1, 2, \dots,$$

where $p^{-1} + q^{-1} = 1$, $\{v_k(x)\}_{k=1}^\infty$ is biorthogonally conjugate system to $\{u_k(x)\}_{k=1}^\infty$ consisting of root functions of the formally conjugate operator $L^* = -\frac{d^2}{dx^2} + \overline{q(x)}$ (i.e. $L^*v_k = \overline{\lambda_k} v_k + \theta_k v_{k+1}$, $\theta_k = 0$ or $\theta_k = 1$); $\|\cdot\|_{p,K} = \|\cdot\|_{L_p(K)}$, $\|\cdot\|_p = \|\cdot\|_{L_p(G)}$.
Denote

$$\sigma_\nu(x, f) = \sum_{\rho_k \leq \nu} f_k u_k(x), \quad \nu > 0, \quad f_k = (f, v_k) = \int_0^1 f(x) \overline{v_k}(x) dx$$

$$\Delta_{\nu s}(f, K) = \|\sigma_\nu(\cdot, f) - S_\nu(\cdot, f)\|_{L_s(K)},$$

$$\omega_p \eta = \sup_{0 < h \leq \eta} \left\{ \int_0^{1-h} |f(x+h) - f(x)|^p dx \right\}^{\frac{1}{p}}; \quad \delta = \max\{2, q\},$$

where $S_\nu(x, f)$ is a partial sum of the trigonometric Fourier series of the function $f(x) \in L_p(G)$.

Definition. If $\Delta_{\nu s}(f, K) \rightarrow 0$ as $\nu \rightarrow +\infty$, then we will say that the expansions of the function $f(x)$ into biorthogonal series with respect to the system $\{u_k(x)\}_{k=1}^\infty$ and into the trigonometric Fourier series equiconvergent in metrics L_s on the compact $K \subset G$ (in case $s = \infty$, it is about a uniform equiconvergence).

We have following theorem prove in this work:

Theorem. Let $q(x) \in L_r(G)$, $r \in [1, \infty]$, and the system $\{u_k(x)\}_{k=1}^\infty$ satisfy the conditions A_p for some fixed p , $p \geq 1$. Then the expansions of an arbitrary function $f(x) \in L_p(G)$ into biorthogonal series with respect to the system $\{u_k(x)\}_{k=1}^\infty$ and into the trigonometric Fourier series are equiconvergent in metrics L_s , $s \in [1, \infty]$, on every compact $K \subset G$ and the following estimates are true for $\nu \geq 2$:

I. If $p > 1$, then

$$\Delta_{\nu s}(f, K) \leq C(K) \begin{cases} \nu^{-1} \|f\|_p + \omega_1(f, \nu^{-1}), & \frac{1}{r} - \frac{1}{s} < \frac{1}{\delta} \\ \nu^{-1} \ln^{1-\frac{1}{\delta}} \nu \|f\|_p + \omega_1(f, \nu^{-1}), & \frac{1}{r} - \frac{1}{s} = \frac{1}{\delta} \\ \nu^{\frac{1}{r}-\frac{1}{s}-\frac{1}{\delta}-1} \|f\|_p + \omega_1(f, \nu^{-1}), & \frac{1}{r} - \frac{1}{s} > \frac{1}{\delta} \end{cases} \quad (1)$$

II. If $p = 1$, then

$$\leq C(K) \begin{cases} \nu^{-1} \|f\|_1 + \omega_1(f, \nu^{-1}), & s < r \\ \nu^{-1} \ln^2 \nu \|f\|_1 + \omega_1(f, \nu^{-1}), & s = r < 2 \text{ or } s = r = \infty \\ \nu^{-1} \ln^{2(1-\frac{1}{r})} \nu \|f\|_1 + \omega_1(f, \nu^{-1}), & 2 \leq s = r < \infty \\ \nu^{\frac{1}{r}-\frac{1}{s}-1} \|f\|_1 + \omega_1(f, \nu^{-1}), & s > r \end{cases} \quad (2)$$

III. If $q(x) \equiv 0$, $p \geq 1$, $s \in [1, \infty]$, then

$$\Delta_{\nu s}(f, K) \leq C(K) \{ \nu^{-1} \|f\|_1 + \omega_1(f, \nu^{-1}) \}, \quad (3)$$

where $C(K)$ does not depend on ν and $f(x)$.

References

- [1] V.A. Il'in, Necessary and sufficient conditions for the basis property and equiconvergence with trigonometric series of spectral decompositions. I (II), Diff.uravn. 16 (1980), no.5 (6), 771-794 (980-1009)
- [2] V.M. Kurbanov, Equiconvergence of biorthogonal expansions in root functions of differential operators I (II), Diff.uravn. 35 (1999) (36 (2000)), no.12 (3), 1619-1633 (319-335)
- [3] V.M. Kurbanov, L.Z. Buksayeva, Estimating the uniform equiconvergence rate for the Schrödinger operator with summable potential on a compact set. Proc.of IMM, 52 (2026), no.1, 163-175

Roughness Behaviors of Fractal Interpolation Functions via Cesàro Mean

Sema Metin, Uğur Değer

Mersin University, Mersin, Türkiye

(e-mail: semametin1@gmail.com, udeger@mersin.edu.tr)

Fractal Interpolation Functions (FIF) are utilized to model irregular shapes where classical Euclidean geometry remains insufficient [1]. In this study, Fractal Interpolation Functions are expressed through Fourier series using both constant and variable scaling factors. This mathematical approach offers the opportunity to approximate irregular signals with complex and rough structures in the closest manner via periodic waves. In the existing literature, variable scaling factors have provided “smoother” and lower-roughness graphs compared to constant factors [2]. In this work, the Cesaro mean was applied to Fractal Interpolation Functions depending on the scaling factors, and the roughness behaviors of the functions were investigated. The obtained data were simulated via MATLAB and illustrated with graphics. The results demonstrate that as the order of the Cesaro mean increases, the degree of roughness decreases. Consequently, this study reveals that Cesàro means are a powerful and reliable analytical tool in the Fourier analysis of fractal functions and in the mathematical control of high-frequency roughness structures.

References

- [1] B.B. Mandelbrot, The Fractal Geometry of Nature, 1st ed., W.H. Freeman, New York, 1982.
- [2] C. Kavitha, A. Gowrisankar, F.A. Rihan, R. Rakkiyappan, Fourier series approximation of fractal functions, Partial Differential Equations in Applied Mathematics, 2025, 13, 101038.

Mean Ergodic Theorem for Probability Measures

Heybetkulu Mustafayev

Azerbaijan State Oil and Industry University, Baku, Azerbaijan
(e-mail: hsmustafayev@yahoo.com)

Let G be a locally compact group with the left Haar measure m_G and let $M(G)$ be the convolution measure algebra of G . As usual, $C_0(G)$ will denote the space of all complex valued continuous functions on G vanishing at infinity. Since $C_0(G)^* = M(G)$, the space $M(G)$ carries the weak* topology $\sigma(M(G), C_0(G))$. In the following, the w^* -topology on $M(G)$ always means this topology. For a subset S of G , by $[S]$ we will denote the closed subgroup of G generated by S . A probability measure $\mu \in M(G)$ is said to be *adapted* if $[supp\mu] = G$. Also, a probability measure $\mu \in M(G)$ is said to be *strictly aperiodic* if the support of μ is not contained in a proper closed left cosets gH ($H \neq G, g \in G \setminus H$) of G .

The classical Kawada-Itô theorem [3, Theorem 7] asserts that if μ is an adapted measure on a compact metrisable group G , then the sequence of probability measures $\left\{ \frac{1}{n+1} \sum_{i=0}^n \mu^i \right\}_{n \in \mathbb{N}}$ weak* converges to the Haar measure on G . If μ is an adapted and strictly aperiodic measure on a compact metrisable group G , then $w^* - \lim_{n \rightarrow \infty} \mu^n = m_G$ [3, Theorem 8]. For related results see also, [1, 2, 5, 6].

It is well known that if G is a compact group, then the (normalized) Haar measure m_G is an idempotent measure on G with $suppm_G = G$. If H is a closed subgroup of G , then the measure m_H may be regarded as a measure on G by putting $\overline{m}_H(B) = m_H(B \cap H)$ for every Borel subset B of G . Notice that $supp\overline{m}_H = H$.

The following result combines Kawada-Itô theorems.

Theorem. *Let μ and ν be two probability measures on a compact group G and assume that ν is strictly aperiodic. Then we have*

$$w^* - \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=0}^n \mu^i * \nu^{n-i} = \overline{m}_{[supp\mu]} * \overline{m}_{[supp\nu]}.$$

References

- [1] J.Galindo and E.Jordá, Ergodic properties of convolution operators, J. Oper. Theory, 86(2021), 469-501.
- [2] U.Grenander, Probabilities on Algebraic Structures, Almqvist&Wiksell, Stockholm, John Wiley&Sons Inc. New York-London, 1968 (in russian).
- [3] Y.Kawada and K.Itô, On the probability distribution on a compact group, Proc. Phys.-Math. Soc. Japan, 22(1940), 977-998.
- [4] H.Mustafayev, Mean ergodic theorems for multipliers on Banach algebras, J. Fourier Anal. Appl. 25(2019), 393-426.
- [5] H.Mustafayev, A note on the Kawada-Itô theorem, Statist. Probab. Lett. 181(2022), Paper no.109261, 6p.

Schauder-Type Estimates for Higher Order Elliptic Equations in Grand Sobolev Spaces

Sabina Sadigova^a, Vafa Alili^b, Asmar Mirzabalayeva^a

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Baku Slavic University, Baku, Azerbaijan*

(e-mail: sadigova.sr@gmail.com, esmer26@mail.ru, alilivefa@mail.ru)

We consider an m -th order elliptic equation whose principal part has continuous coefficients. Let $L_p(\Omega)$, $1 < p < +\infty$, denote the grand Lebesgue space on a bounded domain $\Omega \subset R^n$, and let $W_p^m(\Omega)$ be the corresponding grand Sobolev space. It is proved that, if the boundary $\partial\Omega$ is sufficiently smooth, then, under appropriate conditions on the coefficients of the elliptic operator under consideration, a Schauder-type estimate up to the boundary holds for this operator. Such estimates play an important role in establishing the Fredholmness of boundary value problems for elliptic equations.

This work was supported by the Azerbaijan Science Foundation-Grant No. AEF-MGC-2024-2(50)-16/02/1-M-02

References

- [1] B.T. Bilalov, S.R. Sadigova, *On solvability in the small of higher order elliptic equations in grand-Sobolev spaces*, Complex Variables and Elliptic Equations, 2021, 66(12), 2117–2130.
- [2] B.T. Bilalov, S.R. Sadigova, *Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces*, Sahand Communications in Mathematical Analysis, 2021, 1(2), 129–148.

On Generalized Almost Statistical Convergence of Weight g in Partial Metric Spaces

Ekrem Savaş

Uşak University, Uşak, Türkiye
(e-mail: ekremsavas@yahoo.com)

Following the quite recent development in [1], we present the concept of generalized almost statistical convergence of weight g in partial metric space. We further investigate the relationships between generalized almost statistical convergence of weight g and strong $(\hat{V}, \lambda, \rho)$ -almost summability of weight g and subsequently consider the connections between the spaces $[\hat{V}_p^g, \lambda, \rho, f]$ and $\hat{S}^g(\lambda, \rho)$.

References

- [1] E.Savas, *Some Sequence Spaces of Weight f* , Proceedings of the Seventh International Conference on Mathematics and Computing: ICMC 2021, 2022, 815-822.
- [2] S. G. MATTHEWS, Partial metric topology, Annals of the New York Academy of Sciences, 728, 183-197, 1994. doi: 10.1111/j.1749-6632.1994.tb44144.x.

THEORY OF FUNCTIONS

On the Behavior of m -th ($m \geq 1$) Derivatives of Algebraic Polynomials in Unbounded Domains of the Complex Plane

Fahreddin Abdullayev^a, Meerim Imashkyzy^b

^a*Uşak University, Uşak, Türkiye*

^b*Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan*

(e-mail: fahreddin.abdullayev@usak.edu.tr;

meerim.imashkyzy@manas.edu.kg)

Let $\mathbb{C}$ be a complex plane; $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan domain with boundary $L := \partial G$; $\Omega := \overline{\mathbb{C}} \setminus G = \text{ext}L$. Denote by $\Phi : \Omega \rightarrow \{w : |w| > 1\}$ the univalent conformal mapping normalized by $\Phi(\infty) = \infty$ and $\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z} > 0$.

For $t \geq 1$, we define

$$L_t := \{z : |\Phi(z)| = t\}, \quad L_1 \equiv L, \quad G_t := \text{int}L_t, \quad \Omega_t := \text{ext}L_t.$$

Let us denote by

$$\wp_n := \left\{ P_n(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \mathbb{C}, \deg P_n \leq n, \quad n \in \mathbb{N} \right\}$$

Let $\{z_j\}_{j=1}^l \in L$ be the fixed system of distinct points. For some fixed R_0 , $1 < R_0 < \infty$, consider generalized Jacobi weight function

$$h(z) := \begin{cases} h_0(z) \prod_{j=1}^l |z - z_j|^{\gamma_j}, & z \in \overline{G}_{R_0}, \\ 0, & z \in \mathbb{C} \setminus \overline{G}_{R_0}, \end{cases} \quad (1)$$

where $\gamma_j > -1$, for all $j = 1, 2, \dots, l$, and $h_0(z) \geq c_0(L) > 0$ for some constant $c_0(L) > 0$.

For each $0 < p \leq \infty$ and arbitrary Jordan domain G with rectifiable boundary $L = \partial G$, we introduce:

$$\|P_n\|_p := \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) |P_n(z)|^p |dz| \right)^{1/p} < \infty, 0 < p < \infty;$$

$$\|P_n\|_\infty := \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \quad p = \infty; \quad \mathcal{L}_p(1, L) =: \mathcal{L}_p(L).$$

The well-known Bernstein-Walsh inequality [1] states that:

$$\|P_n\|_{C(\overline{G}_\rho)} \leq \rho^n \|P_n\|_{C(\overline{G})}, \quad \forall P_n \in \wp_n, \quad \rho > 1. \quad (2)$$

Subsequently, in [2] an analogue of this inequality was found for the space $\mathcal{L}_p(L)$, $p > 0$:

$$\|P_n\|_{\mathcal{L}_p(L_\rho)} \leq \rho^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \quad \forall P_n \in \wp_n. \quad (3)$$

Next, the estimate (3) was transferred to the weighted $\mathcal{L}_p(h, L_R)$ space with the weight function $h(z) \neq 1$ determined by formula (1), as follows [3, Lemma 2.4]:

$$\|P_n\|_{\mathcal{L}_p(h, L_\rho)} \leq \rho^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \quad \gamma^* = \max \{0; \gamma_j : 1 \leq j \leq l\}. \quad (4)$$

In order to give two-dimensional analogues of inequalities (2)- (4), it is necessary to give the following definitions.

Now let's introduce the quantity:

$$\begin{aligned} \|P_n\|_{A_p(h,G)} &:= \left(\iint_G h(z) |P_n(z)|^p d\sigma_z \right)^{1/p}, \quad 0 < p < \infty, \\ \|P_n\|_{A_\infty(1,G)} &:= \max_{z \in \overline{G}} |P_n(z)|, \quad p = \infty; \quad A_p(1, G) =: A_p(G), \end{aligned}$$

where σ be the two-dimensional Lebesgue measure.

The similar estimate for an arbitrary Jordan domain G and $h(z) \equiv 1$ was given in the [4, Theorem 1.1] as follows:

$$\|P_n\|_{A_p(G_\rho)} \leq c_3 \rho^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{1+\frac{1}{n}})}, \quad \rho > 1 + \frac{1}{n}, \quad p > 0, \quad (5)$$

where $c_3 = \left(\frac{2}{e^p-1}\right)^{\frac{1}{p}} \left[1 + O\left(\frac{1}{n}\right)\right]$, $n \rightarrow \infty$, is asymptotically sharp constant.

On the other hand, in [5] for domains bounded by a rectifiable quasicircle, a local estimate of type (5) was found, which also takes into account the position of a point outside the domain, of the following type:

$$|P_n(z)| \leq c(L) \frac{\sqrt{n}}{d(z, L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega,$$

where $c(L) > 0$ is a constant depending only of L and $d(z, L) := \text{dist}(z, L) = \inf \{|\zeta - z| : \zeta \in L\}$.

In this we continue to study pointwise estimation in unbounded domains of complex plane in the following form:

$$|P_n^{(m)}(z)| \leq \eta_n(G, h, p, m, z) \|P_n\|_X, \quad z \in \Omega,$$

for the cases $X = \mathcal{L}_p(h, L)$ and $X = A_p(h, G)$, where $\eta_n(\cdot) \rightarrow \infty$, as $n \rightarrow \infty$, depending on the properties of the G, h .

References

- [1] J.L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, AMS, Rhode Island, 1960.
- [2] E. Hille, G. Szegő, J.D. Tamarkin, On some generalization of a theorem of A.Markoff, Duke Math. J., 3 (1937) 729-739.
- [3] F.G. Abdullayev, P. Özkartepe, On the growth of algebraic polynomials in the whole complex plane, J. of the Korean Math. Soc. 52 (4) (2015) 699-725.
- [4] F.G. Abdullayev, P. Özkartepe, An analogue of the Bernstein-Walsh lemma in Jordan regions of the complex plane, J. of Ineq. and Appl. 570 (2013) 7 p.
- [5] N. Stylianopoulos, Strong asymptotics for Bergman polynomials over domains with corners and applications, Const. Approx., 38 (1) (2012) 59-100.

Lower estimates for the best approximation of continuous holomorphic functions

Fahreddin Abdullayev^a, Viktor Savchuk^b

^a*Usak University, 4000 Usak, Türkiye*

^b*Institute of Mathematics of NAS of Ukraine, Kyiv 01004, Ukraine*

(e-mail: fahreddin.abdullayev@usak.edu.tr, savchuk@imath.kiev.ua)

This talk is devoted to exact lower bounds for the best approximation of functions in the disk algebra $A(\overline{\mathbb{D}})$, the space of all functions that are holomorphic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and continuous on its closure $\overline{\mathbb{D}}$, equipped with the norm $\|f\| = \max_{z \in \overline{\mathbb{D}}} |f(z)|$.

Let f be holomorphic function in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let

$$f(z) = \sum_{k=0}^{\infty} \widehat{f}_k z^k, \quad \widehat{f}_k := \frac{f^{(k)}(0)}{k!},$$

be its Taylor series.

The best approximation of $f \in H^q$ is the quantity

$$E_n(f) := \inf_{P_{n-1} \in \mathcal{P}_{n-1}} \|f - P_{n-1}\|, \quad n \in \mathbb{N},$$

where $\mathcal{P}_{n-1}$ is the set of all algebraic polynomials of degree at most $n - 1$.

Theorem 1. *Let $n, N \in \mathbb{N}$, $n \leq N$. Then for any $f \in A(\overline{\mathbb{D}})$, we have*

$$\frac{1}{2} \left(|\widehat{f}_N| + \sqrt{|\widehat{f}_N|^2 + 4 \sum_{k=n}^{[(N-1)/2]} |\widehat{f}_k|^2} \right) \leq E_n(f),$$

where $[\cdot]$ denotes the entire part of number, and the sum is empty if $[(N - 1)/2] < n$.

Equality holds for any function $f \in A(\overline{\mathbb{D}})$ such that the polynomial

$$1 + \lambda \sum_{k=n}^{[(N-1)/2]} \widehat{f}_k t^{N-k}, \quad t \in \mathbb{C},$$

where

$$\lambda = \frac{1}{2 \sum_{k=n}^{[(N-1)/2]} |\hat{f}_k|^2} \left(\sqrt{|\hat{f}_N|^2 + 4 \sum_{k=n}^{[(N-1)/2]} |\hat{f}_k|^2} - |\hat{f}_N| \right),$$

has no zeros in $\mathbb{D}$.

On the Generalization of Sierpiński's Inequality

Yagub Aliyev^a, Zamina Guliyeva^b

^a*School of IT and Engineering, ADA University, Ahmadbey Aghaoglu str. 61, Baku, Azerbaijan*

^b*School of IT and Engineering, ADA University, Ahmadbey Aghaoglu str. 61, Baku, Azerbaijan*

(e-mail: yaliyev@ada.edu.az, zguliyeva@ada.edu.az)

In this work, we investigate an extension of the classical inequality established by Sierpiński in [3]. For any set of n positive real numbers, let A_n , G_n , and H_n denote their arithmetic, geometric, and harmonic means, respectively. The fundamental relationship can be alternatively framed through log-transformed ratios, leading to the following formulation:

$$\frac{1}{n-1} \leq \frac{\ln G_n - \ln H_n}{\ln A_n - \ln G_n} \leq n-1. \quad (1)$$

Our main focus centers on establishing that the boundary values $1/(n-1)$ and $n-1$ expressed in the above inequalities constitute the sharpest possible constants. By demonstrating this optimality, we verify that the boundaries of the generalized Sierpiński inequality cannot be further restricted or improved.

To evaluate these extreme cases and substantiate the sharp nature of these constants, we systematically employ the optimization method and mathematical techniques originally introduced and developed in [1] and [2].

References

- [1] Y.N. Aliyev, *The best constant for inequality involving the sum of the reciprocals and product of positive numbers with unit sum*, Journal of Inequalities and Applications, 2024, 2024(29), 1–13.
- [2] Y.N. Aliyev, *The Extremal Values of the Ratio of Differences of Means*, Mediterranean Journal of Mathematics, 2025, 22(163), 1–10.
- [3] W. Sierpinski, *Sur une inégalité pour la moyenne arithmétique, géométrique, et harmonique*, Warsch. Sitzungsber, 1909, 2, 354–357.

On the Approximation of Special Functions by Branched Continued Fractions

Roman Dmytryshyn

57 Shevchenko Str., 76018, Ivano-Frankivsk, Ukraine
(e-mail: roman.dmytryshyn@cnu.edu.ua)

Special functions appear quite naturally in almost all fields of science and engineering. Many of these functions are hypergeometric functions or generalizations of them, or can be expressed in terms of them.

We consider the problems of approximating the double hypergeometric functions by branched continued fractions, namely: constructing the simplest fractional structure and investigating its convergence and numerical stability. We have proposed new methods for investigating the convergence and establishing truncation errors bounds and numerical stability sets of the expansions of the double hypergeometric functions, including the well-known Appell hypergeometric series [1, 2], into branched continued fractions.

References

- [1] R. Dmytryshyn, *On the Analytic Continuation of Appell's Hypergeometric Function F_2 to Some Symmetric Domains in the Space $\mathbb{C}^2$* , Symmetry, 2024, 16(11), 1480.

- [2] R. Dmytryshyn, Truncation Error Bounds for Branched Continued Fraction Expansions of Some Appell's Hypergeometric Functions F_2 , *Symmetry*, 2025, 17(8), 1204.

On the Stability to Perturbations of S -fractions

Marta Dmytryshyn^a, Volodymyr Hladun^b

^a11 Lvivska Str., 46009, Ternopil, Ukraine

^b12 Stepan Bandera Str., 79013, Lviv, Ukraine

(e-mail: m.dmytryshyn@wunu.edu.ua, volodymyr.r.hladun@lpnu.ua)

Continued fractions are one of the most efficient approximation tools, having natural properties such as wide convergence regions, high convergence speed, and computational stability.

We consider the problem of stability to perturbations of functional continued fractions [1], in particular, S -fractions (Stieltjes continued fractions). An analytical approach is proposed that allows for the evaluation of the influence of perturbations of the coefficients of continued fraction and its complex variable on the value of its approximant. Recurrence relations for the relative errors of the approximants are established and presented in the form of a continued fraction. Sufficient conditions for stability to perturbations are obtained, as well as explicit estimates for the relative errors of the approximants. Sets of stability to perturbations are constructed, in particular, semi-circular and circular sets in the complex plane [2].

References

- [1] Dmytryshyn M., Hladun V. *On the Sets of Stability to Perturbations of Some Continued Fraction with Applications*, *Symmetry*, 2025, 17(9), 1442.
- [2] Hladun V., Dmytryshyn M. *On the Stability to Perturbations of Stieltjes Continued Fractions with Complex Elements*, *Carpathian Math. Publ.*, 2025, 17(2), 565–578.

The Geometry of $A(z)$ -Analytic Spaces and the Bergman Reproducing Kernel

Behzod Husenov

Muhammad Ikbal str. 11, 200117, Bukhara, Uzbekistan
(e-mail: b.e.husenov@buxdu.uz)

As is well known, the solution of the Beltrami equation

$$\bar{D}_A f = \frac{\partial f}{\partial \bar{z}} - A(z) \frac{\partial f}{\partial z} = 0 \quad (1)$$

is directly related to quasiconformal mappings. In the general case, with respect to the function $A(z)$, it is assumed that it is measurable and $|A(z)| \leq c$ almost everywhere in the domain under consideration $D \subset \mathbb{C}$, where $c < 1$. Solutions of equation (1) are called $A(z)$ -analytic functions. The most interesting case is when $A(z)$ is an antiholomorphic function in the domain D . Then, according to (1), the class of $A(z)$ -analytic functions $f \in \mathcal{O}_A(D)$ is characterized by the property that $\bar{D}_A f = 0$. Since an anti-analytic function is infinitely smooth, it follows from I. N. Vekua's theorem that $\mathcal{O}_A(D) \subset C^\infty(D)$. Suppose now that a domain $D \subset \mathbb{C}$ is convex and $\xi \in D$ is its fixed point. Consider the kernel

$$K(z, \xi) = \frac{1}{2\pi i} \frac{1}{z - \xi + \int_{\gamma(\xi, z)} \overline{A(\tau)} d\tau}, \quad (2)$$

where $\gamma(\xi, z)$ is a smooth curve connecting the points $\xi, z \in D$. Since the domain D is simply connected and the function $\bar{A}(z)$ is holomorphic, the integral $I(z) = \int_{\gamma(\xi, z)} \overline{A(\tau)} d\tau$ does not depend on the integration path; it coincides with

the primitive, $I'(z) = \bar{A}(z)$ (see [2]).

Theorem 1. (see [1]). *$K(z, \xi)$ is an $A(z)$ -analytic function outside of the point $z = \xi$, i.e. $K(z, \xi) \in \mathcal{O}_A(D \setminus \{\xi\})$. Moreover, at $z = \xi$ the function $K(z, \xi)$ has a simple pole.*

Note that if the domain $D \subset \mathbb{C}$ is not convex, but only simply connected, then although the function $\psi(z, \xi) = z - \xi + \int_{\gamma(\xi, z)} \overline{A(\tau)} d\tau$, is uniquely defined

in the domain D , but a priori, it can have other isolated zeros besides $\xi : \psi(\xi, z) = 0, z \in P \setminus \{\xi, \xi_1, \xi_2, \dots\}$. In this regard, below we consider mainly the class of $A(z)$ -analytic functions in convex domains $D \subset \mathbb{C}$.

Let the domain $D \subset \mathbb{C}$ be convex. According to Theorem 1, the function $\psi(\xi, z) \in O_A(D)$ performs an internal mapping. In particular, the set

$$L(a, r) = \{z \in D : |\psi(z, a)| < r\}$$

for sufficiently small $r > 0$ is an open set compactly belonging to D , and it contains a point a . This set is called $A(z)$ -lemniscate, centered at point a and denoted as $L(a, r)$ (see [1]).

As an example, we note that for multiply connected domains (for instance, an annulus), the Bergman kernel may have zeros off the diagonal; this is a well-known phenomenon and highlights the dependence of the kernel's subtle properties on the topology of the domain:

$$K_A(z, \zeta) = \frac{1}{\pi} \left[\sum_{n=0}^{\infty} \frac{n+1}{1-r^{2n+2}} (z\bar{\zeta})^n + \sum_{n=0}^{\infty} \frac{(n+1)r^{2n+2}}{1-r^{2n+2}} (\bar{z}\zeta)^{n-2} \right].$$

Unlike the disk, for an annulus this kernel may have zeros off the diagonal $z = \zeta$.

As an example, consider $r = \frac{1}{2}$. Owing to circular symmetry, the kernel depends only on $t = z\bar{\zeta}$. Let us set $\zeta = \rho > 0$ and $z = -\rho$ (on the same circle, but in opposite directions); then $t = -\rho^2 < 0$. For $\rho \approx 0.5029301539$, we obtain

$$K_{A_{\frac{1}{2}}}(-\rho, \rho) = 0$$

(numerically, with accuracy of order 10^{-13}). That is, the kernel has a zero at $z \neq \zeta$.

Let us analyze step by step why the Bergman (or Szego) kernel is unique, and how it follows from this that the counterexample indeed demonstrates the existence of a unique kernel with zeros off the diagonal, rather than the existence of "other kernels."

1. Why the kernel is unique

Let $H \subset L^2(D)$ be a Hilbert space of holomorphic functions on a domain D . The Bergman kernel $K(z, \zeta)$ is defined in H by the reproducing property:

$$f(z) = \int_D f(\zeta) K(z, \zeta) dA(\zeta).$$

From the general theory of reproducing kernels (the Riesz—Aronszajn theorem), it follows that:

If the Hilbert space H is given, then the corresponding reproducing kernel $K(z, \zeta)$ is unique.

That is, no other function $K(z, \zeta)$ with the same reproducing property can exist—only one such K corresponds to the given Hilbert space.

2. When one says that “the kernel has other zeros,” this does not mean that there exists a different kernel.

What is meant is that this unique kernel (unique by definition) may have points (z, ζ) , distinct from the diagonal $z = \zeta$, at which $K(z, \zeta) = 0$.

In the unit disk $K_D(z, \zeta) = \frac{1}{\pi}(1 - z\bar{\zeta})^{-2}$, the Bergman kernel does not vanish anywhere except in the trivial case, since the denominator cannot be zero inside the disk.

However, in ring-type domains or multiply connected domains, the structure of the basis becomes more complicated, and the combination of positive and negative powers of z^n, z^{-n-1} generates sign-alternating series, which leads to the possible occurrence of zeros off the diagonal.

3. Example (a unique kernel, but with a zero off the diagonal). For the ring $A_r = \{r < |z| < 1\}$, the following equality holds:

$$K_A(z, \zeta) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{(n+1)z^n \bar{\zeta}^n}{1 - r^{2(n+1)}}.$$

This is the unique kernel of the space $A^2(A_r)$.

References

- [1] A.Sadullayev, N.M.Zhabborov, *On a class of A -analytic functions*, Journal Siberian Federal University. Mathematics and Physics, 2016, 9(3), 374-383.
- [2] N.M.Zhabborov, T.U.Otaboyev, *An analogue of the Cauchy integral formula for $A(z)$ -analytic functions*, Uzbek Mathematical Journal, 2016, 4, 50-59.

Approximation Properties of Feedforward Neural Networks on Topological Spaces

Vugar Ismailov

Institute of Mathematics, Baku, Azerbaijan
(e-mail: vugaris@gmail.com)

The universal approximation theorem is one of the fundamental results in neural network theory. It states that feedforward neural networks can approximate arbitrary continuous functions on compact subsets of Euclidean spaces with arbitrary accuracy. For a survey article and a monograph on approximation by multilayer perceptron models, we refer to [2, 3].

Classical shallow feedforward neural networks are based on functions of the form

$$\sum_{i=1}^r c_i \sigma(w_i \cdot x - \theta_i), \quad x \in \mathbb{R}^d,$$

where σ is a fixed activation function and $w_i \cdot x$ are continuous linear functionals. In [1], a more general framework was introduced in which the input space is an arbitrary topological space.

Let X be a topological space and let $\mathcal{A}(X) \subset C(X)$ be a prescribed family of continuous functions. Consider the class

$$\mathcal{N}(\sigma) = \text{span} \{ \sigma(wf(x) - \theta) : f \in \mathcal{A}(X), w, \theta \in \mathbb{R} \}.$$

We say that $\mathcal{A}(X)$ possesses the D -property if

$$\text{span} \{ u \circ f : u \in C(\mathbb{R}), f \in \mathcal{A}(X) \}$$

is dense in $C(X)$ in the topology of uniform convergence on compact sets.

Recall that an activation function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is called a Tauber–Wiener function if

$$\text{span} \{ \sigma(wx - \theta) : w, \theta \in \mathbb{R} \}$$

is dense in $C[a, b]$ for every compact interval $[a, b] \subset \mathbb{R}$. For example, it follows from the theorem of Leshno, Lin, Pinkus and Schocken [4] that every continuous nonpolynomial activation function is Tauber–Wiener.

The following theorem holds.

Theorem 1. *Let X be a topological space, let $\mathcal{A}(X)$ possess the D -property, and let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a Tauber–Wiener activation function. Then for every function $g \in C(X)$, every compact set $K \subset X$, and every $\varepsilon > 0$, there exist functions $f_i \in \mathcal{A}(X)$ and numbers $c_i, w_i, \theta_i \in \mathbb{R}$ such that*

$$\max_{x \in K} \left| g(x) - \sum_{i=1}^r c_i \sigma(w_i f_i(x) - \theta_i) \right| < \varepsilon.$$

This theorem shows that the universal approximation property is essentially topological and does not require linear, metric, normed, or completeness structures on the input space.

An important special case is obtained when X is a locally convex topological vector space and $\mathcal{A}(X) = X^*$, where X^* denotes the continuous dual space.

Theorem 2. *Let X be a locally convex topological vector space and let σ be a continuous nonpolynomial activation function. Then for every compact set $K \subset X$, every function $g \in C(K)$ and every $\varepsilon > 0$, there exist functionals $f_i \in X^*$ and numbers $c_i, \theta_i \in \mathbb{R}$ such that*

$$\max_{x \in K} \left| g(x) - \sum_{i=1}^r c_i \sigma(f_i(x) - \theta_i) \right| < \varepsilon.$$

The proof combines the Hahn–Banach and Stone–Weierstrass theorems with the classical theorem of Leshno, Lin, Pinkus and Schocken [4]. The general approximation theorem also has applications to superposition theory. In particular, combining it with Ostrand’s extension of Kolmogorov’s superposition theorem yields approximate neural network representation for continuous functions on compact metric spaces by means of universal inner functions and a single smooth activation function. For details and proofs, we refer to [1].

References

- [1] A. Pinkus, *Approximation theory of the MLP model in neural networks*, Acta Numerica, 1999, 8, 143–195.
- [2] V.E. Ismailov, *Ridge Functions and Applications in Neural Networks*, Mathematical Surveys and Monographs, 263, American Mathematical Society, 2021.

- [3] V.E. Ismailov, *On shallow feedforward neural networks with inputs from a topological space*, Annals of Mathematics and Artificial Intelligence, 2026. <https://doi.org/10.1007/s10472-026-10003-7>
- [4] M. Leshno, V. Ya. Lin, A. Pinkus and S. Schocken, *Multilayer feedforward networks with a nonpolynomial activation function can approximate any function*, Neural Networks **6** (1993), 861-867.

Approximation Theorems in Variable Exponent Smirnov Classes

Daniyal Israfilov

*Institute of Mathematics, Baku, Azerbaijan
Balikesir University, Balikesir, Türkiye
(e-mail: mdaniyal@balikesir.edu.tr)*

In this talk, we discuss approximation problems in variable exponent Smirnov classes of analytic functions defined on bounded simply connected domains of the complex plane.

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$, and let $p(\cdot): \Gamma \rightarrow [1, \infty)$ be a Lebesgue measurable function. The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ is defined as the set of all measurable functions f defined on Γ , such that $\int_{\Gamma} |f(z)|^{p(z)} |dz| < \infty$. If $p^+ := \text{ess sup}_{z \in \Gamma} p(z) < \infty$, then $L^{p(\cdot)}(\Gamma)$ is a Banach space, equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf \left\{ \lambda > 0 : \int_{\Gamma} |f(z)/\lambda|^{p(z)} |dz| \leq 1 \right\} < \infty.$$

Definition 1 We say that $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$, if $1 < p_- := \text{ess inf}_{z \in \Gamma} p(z) \leq \text{ess sup}_{z \in \Gamma} p(z) =: p^+ < \infty$, and in addition,

$$|p(z_1) - p(z_2)| \ln \left(\frac{|\Gamma|}{|z_1 - z_2|} \right) \leq c(p), \quad \forall z_1, z_2 \in \Gamma \text{ and } z_1 \neq z_2,$$

where $c(p) > 0$ is a constant and $|\Gamma|$ denotes the Lebesgue measure of Γ .

Let $G \subset \mathbb{C}$ be a simply connected domain bounded by a rectifiable Jordan curve Γ , and let $G^- := \overline{\mathbb{C}} \setminus \overline{G}$. We also denote $\mathbb{U} := \{w \in \mathbb{C} : |w| < 1\}$ and $\mathbb{U}^- := \overline{\mathbb{C}} \setminus \overline{\mathbb{U}}$.

By $E_p(G)$, $1 \leq p < \infty$, we denote the well known Smirnov class of analytic in G function. Every function $f \in E_p(G)$ admits non-tangential boundary values a.e. on Γ , and the boundary function belongs to $L_p(\Gamma)$. Considering the large class $E_1(G)$ and the variable exponent Lebesgue spaces $L_{p(\cdot)}(\Gamma)$, $1 \leq p(\cdot) < \infty$, the variable exponent Smirnov class $E_{p(\cdot)}(G)$ is defined as $E_{p(\cdot)}(G) := \{f \in E_1(G) : f|_{\Gamma} \in L_{p(\cdot)}(\Gamma)\}$. If $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$, then the class $E_{p(\cdot)}(G)$ is a Banach space, endowed with the norm $\|f\|_{E_{p(\cdot)}(G)} := \|f\|_{L_{p(\cdot)}(\Gamma)}$.

From now on we assume that the boundary curve Γ is Dini-smooth, defined as follows.

Definition 2 A Jordan curve Γ is called Dini-smooth, if

$$\int_0^\delta \omega(\theta, s) / s ds < \infty, \quad \delta > 0,$$

where $\theta(s)$ denotes the angle, between the tangent to Γ and the positive real axis expressed as a function of arc length s , and $\omega(\theta, s) := \sup_{|t_1 - t_2| \leq s} \{|\theta(t_1) - \theta(t_2)|\}$.

The class of Dini-smooth curves is denoted by D .

Let ψ denotes the conformal mapping of the exterior domain $\mathbb{U}^-$ onto G^- , normalized by the conditions: $\psi(\infty) = \infty$, $\lim_{w \rightarrow \infty} \psi(w)/w > 0$. For $f \in L_{p(\cdot)}(\Gamma)$, $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$ and $w \in \gamma$, we define: $p_0(w) := (p \circ \psi)(w)$, $f_0(w) := (f \circ \psi)(w)$.

Assume that f is analytic in G and $f^{(r)} \in E_{p(\cdot)}(G)$, $r = 0, 1, 2, \dots$. We set: $f_0^{(r)}(w) := f^{(r)}[\psi(w)]$, $r \in N$, and define the modulus of smoothness of $f^{(r)}$, as $\Omega(f^{(r)}, 1/n)_{p(\cdot)} := \sup_{0 < |h| \leq \delta} \|f_0^{(r)} - \sigma_h(f_0^{(r)})\|_{L_{p(\cdot)}(\mathbb{T})}$, where

$$\sigma_h(f_0^{(r)})(w) := \frac{1}{h} \int_0^h f_0^{(r)}(we^{it}) dt, \quad w \in \gamma, \quad 0 < h < \pi.$$

Our main results can be formulated as follows:

Theorem 1. Let $G \subset \mathbb{C}$ be a simply connected domain bounded by a Dini-smooth curve Γ , and let $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$. Assume that f is analytic in

G and $f^{(r)} \in E_{p(\cdot)}(G)$, $r = 0, 1, 2, \dots$. Then, for every $n \in \mathbb{N}$ there exists an algebraic polynomial P_n of degree at most n , such that $\|f - P_n\|_{L_{p(\cdot)}(\Gamma)} \leq \frac{c}{n^r} \Omega(f^{(r)}, 1/n)_{p(\cdot)}$, where $c = c(\Gamma, p) > 0$ is a constant nondependent of n .

Theorem 2 Let $f \in L^{p(\cdot)}(\Gamma)$, $\Gamma \in D$, $p(\cdot) \in P_{\log}(\Gamma)$ and $\alpha \in (0, 1]$. Suppose that for some $r \in \mathbb{N}$, and every $n \in \mathbb{N}$ there exists an algebraic polynomial P_n of degree n such that $\|f - P_n\|_{L_{p(\cdot)}(\Gamma)} \leq c/n^{r+\alpha}$, with some constant $c = c(\Gamma, p) > 0$, nondependent of n . Then there exists a function (denoted again by) f , analytic in G , whose non tangential boundary values coincide with the given boundary function f a.e. on Γ . Moreover, $f^{(r)} \in E_{p(\cdot)}(G)$ and its modulus of smoothness $\Omega(f^{(r)}, 1/n)_{p(\cdot)}$ satisfies the inequalities:

$$\begin{aligned} 1. \Omega(f^{(r)}, 1/n)_{p(\cdot)} &\leq c/n^\alpha, \text{ if } 0 < \alpha < 1, \\ 2. \Omega(f^{(r)}, 1/n)_{p(\cdot)} &\leq c \frac{\ln(n+1)}{n}, \text{ if } \alpha = 1. \end{aligned}$$

For $f \in E_{p(\cdot)}(G)$ we define the best polynomial approximation number by $E_n^{p(\cdot)}(f, G) := \inf_{p_n \in P_n} \|f - p_n\|_{L_{p(\cdot)}(\Gamma)}$, where P_n is the set of all algebraic polynomials of degree at most n . Combining *Theorems 1 and 2* we have

Theorem 3 Let $\Gamma \in D$, $p(\cdot) \in P_{\log}(\Gamma)$, $r \in \{0, 1, 2, \dots\}$, and $0 < \alpha < 1$. The statements: 1. $E_n^{p(\cdot)}(f, G) \leq c/n^{r+\alpha}$, and 2. $f^{(r)} \in E_{p(\cdot)}(G)$ and $\Omega(f^{(r)}, 1/n)_{p(\cdot)} < c/n^\alpha$ are equivalent.

When $p(\cdot) = p = \text{const}$, *Theorems 1-3*, formulated in terms of the classical modulus of smoothness, constructed via usual shift operator, were obtained by S. Ya. Alper [1]. Analogous results for more general domains in the case of *constant exponents* in particular were established in [2, 3].

This work was made under the project "Approximation problems in weighted variable exponent Smirnov classes" supported by Balikesir University grant Nr: 2025/237 [D6].

References

- [1] S. Ya. Alper, *Approximation in mean of analytic functions of class E_p* (Russian), In : Investigations on the modern problems of the function theory of a complex variable, Moscow: Gos. Izdat. Fiz.-Mat. Lit. 1960: 273-286.

- [2] I. I. Ibragimov, D. I. Mamedkhanov, *A constructive characterization of a certain class of functions*. Dokl. Akad. Nauk SSSR, 1975, 223, 35-37, Soviet Math. Dokl., 1976, 4, 820-823.
- [3] D. M. Israfilov, A. Guven, *Approximation in weighted Smirnov classes*, East Journal on Approximation, 2005, 11, 91-102.

Comonotone Approximation by Periodic Splines of Minimal Defect

Kirill Kopotun

University of Manitoba, Winnipeg, Canada
(e-mail: Kirill.Kopotun@umanitoba.ca)

Let $\tilde{C}^r$, $r \in \mathbb{N}_0$, be the class of r times continuously differentiable 1-periodic functions $f : \mathbb{R} \rightarrow \mathbb{R}$. In this talk, I discuss the exact estimates in terms of the k -th moduli of smoothness of $f^{(r)}$, of errors of comonotone approximation of functions in $\tilde{C}^r$ by periodic splines of minimal defect.

In particular, let $\tilde{W}^r$, $r \in \mathbb{N}$, be the space of 1-periodic functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f^{(r-1)} \in AC(\mathbb{R})$ and $f^{(r)} \in L_\infty(\mathbb{R})$, and denote by $\tilde{\Sigma}_m(\mathbf{u}_n)$ the space of 1-periodic splines of order m (degree $m - 1$) of minimum defect (highest smoothness) with the knots $\mathbf{u}_n = (j/n)_{j \in \mathbb{Z}}$.

One of the corollaries illustrating the main results is the following statement:

If a function $f \in \tilde{W}^r$, $r \in \mathbb{N}$, has $2s \in \mathbb{N}$ changes of its monotonicity within a period, then, for each $n \in \mathbb{N}$ and each $m \geq r$, there is a spline of minimal defect $\tilde{\Sigma}_m(\mathbf{u}_n)$ which is comonotone with f (i.e., $f'(x)S'(x) \geq 0$ a.e. in $\mathbb{R}$) and such that

$$\|f - S\|_{C(\mathbb{R})} \leq cn^{-r} \|f^{(r)}\|_{L_\infty(\mathbb{R})},$$

where c is a constant that depends only on m , r and s .

(This is a joint work with D. Leviatan and I. Shevchuk.)

On the Analytical Extension of Some Ratios of Triple Hypergeometric Functions F_M

Ivan Nyzhnyk

Vasyl Stefanyk Carpathian National University, Ivano-Frankivsk, Ukraine
(e-mail: ivan.nyzhnyk.19@pnu.edu.ua)

The Lauricella–Saran’s hypergeometric functions F_M are defined by:

$$F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma_1, \gamma_2; \mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{(\alpha_1)_p (\alpha_2)_{q+r} (\beta_1)_{p+r} (\beta_2)_q}{(\gamma_1)_p (\gamma_2)_{q+r}} \frac{z_1^p z_2^q z_3^r}{p!q!r!},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}$, $\gamma_1, \gamma_2 \notin \{0, -1, -2, \dots\}$, $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{C}^3$, $(\cdot)_k$ is the Pochhammer symbol.

Let $\mathfrak{J} = \{1, 2\}$ and $\mathfrak{J}_k = \{i(k) = (i_0, i_1, i_2, \dots, i_k) : i_r \in \mathfrak{J}, 0 \leq r \leq k\}$, $k \geq 1$. For each $i_0 \in \mathfrak{J}$ the function

$$\frac{(1 - \delta_{i_0}^2 z_1) F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})}{F_M(\alpha_1, \alpha_2 + 1, \beta_1 + \delta_{i_0}^1, \beta_2 + \delta_{i_0}^2; \alpha_1, \gamma_2 + 1; \mathbf{z})}, \quad (1)$$

where $\alpha_1, \gamma_2 \notin \{0, -1, -2, \dots\}$ and δ_i^j is the Kronecker symbol, has a formal branched continued fraction [1]

$$v_{i_0}(z) + \sum_{i_1=1}^2 \frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z})} + \sum_{i_2=1}^2 \frac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z})} + \dots, \quad (2)$$

where

$$v_{i_0}(\mathbf{z}) = 1 - z_1 - \frac{\alpha_2 + \beta_{i_0} + 1}{\gamma_2} (1 - \delta_{i_0}^2 z_1) z_{4-i_0} - \frac{\beta_{3-i_0}}{\gamma_2} (1 - \delta_{i_0}^1 z_1) z_{1+i_0}$$

and for $i(k) \in \mathfrak{J}_k$ and $k \geq 1$

$$u_{i(k)}(\mathbf{z}) = \frac{(\alpha_2 + k)(\beta_{i_k} + \sum_{r=0}^{k-1} \delta_{i_r}^{i_k})}{(\gamma_2 + k - 1)(\gamma_2 + k)} (1 - \delta_{i_k}^2 z_1)^2 z_{4-i_k} (1 - \delta_{i_k}^1 z_1 - z_{4-i_k}),$$

$$v_{i(k)}(\mathbf{z}) = 1 - z_1 - \frac{\alpha_2 + \beta_{i_k} + k + 1 + \sum_{r=0}^{k-1} \delta_{i_r}^{i_k}}{\gamma_2 + k} (1 - \delta_{i_k}^2 z_1) z_{4-i_k} -$$

$$-\frac{\beta_{3-i_k} + \sum_{r=0}^{k-1} \delta_{i_r}^{3-i_k}}{\gamma_2 + k} (1 - \delta_{i_k}^1 z_1) z_{1+i_k}.$$

Theorem. Let $\alpha_1 \in \mathbb{C}$, $\alpha_2, \beta_1, \beta_2, \gamma_2 \in \mathbb{R}$, $\alpha_1 \notin \{0, -1, -2, \dots\}$, $0 < \alpha_2 + 1 \leq \gamma_2$, $0 \leq \beta_{i_0} \leq \gamma_2$, $\beta_1^2 + \beta_2^2 \neq 0$, $i_0 \in \mathfrak{I}$. Then for each $i_0 \in \mathfrak{I}$: 1) the branched continued fraction (2) converges uniformly on every compact subset of the domain

$$\mathfrak{Q}_\eta = \bigcup_{-\pi/2 < \alpha < \pi/2} \mathfrak{Q}_{\eta, \alpha}, \quad 0 < \eta < 1, \quad (3)$$

$$\begin{aligned} \mathfrak{Q}_{\eta, \alpha} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \operatorname{Re}((z_1 + 2(1 - \delta_k^2 z_1)z_{4-k} + (1 - \delta_k^1 z_1)z_{1+k})e^{-i\alpha}) < \eta \cos \alpha, \quad k = 1, 2, \right. \\ \left. 2 \sum_{k=1}^2 (|(1 - \delta_k^2 z_1)^2 z_{4-k}(1 - \delta_k^1 z_1 - z_{4-k})| - \operatorname{Re}((1 - \delta_k^2 z_1)^2 z_{4-k}(1 - \delta_k^1 z_1 - z_{4-k})e^{-2i\alpha})) \right. \\ \left. < (1 - \eta)^2 \cos^2 \alpha \right\}, \quad (4) \end{aligned}$$

to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in $\mathfrak{Q}_\eta$; 2) the function $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of (1) in the domain $\mathfrak{Q}_\eta$.

Corollary. Let $i_0 = 1$, $\alpha_1 \in \mathbb{C}$, $\beta_2, \gamma_2 \in \mathbb{R}$, $\alpha_1 \notin \{0, -1, -2, \dots\}$, $0 < \beta_2 \leq \gamma_2 - 1$, $\gamma_2 \geq 2$. Then the branched continued fraction

$$\frac{1}{v_{i_0}(\mathbf{z})} + \sum_{i_1=1}^2 \frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z})} + \sum_{i_2=1}^2 \frac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z})} + \dots, \quad (5)$$

whose elements are obtained from v_{i_0} , $u_{i(k)}$, and $v_{i(k)}$ by setting $\alpha_2 = \beta_1 = 0$ and replacing γ_2 by $\gamma_2 - 1$, converges uniformly on every compact subset of the domain

$$\mathfrak{H}_\eta = \bigcup_{-\pi/2 < \alpha < \pi/2} \left(\mathfrak{Q}_{\eta, \alpha} \cap \mathfrak{H}_{\eta, \alpha} \right), \quad 0 < \eta < 1, \quad (6)$$

where $\mathfrak{Q}_{\eta, \alpha}$ is defined by (4) and

$$\mathfrak{H}_{\eta, \alpha} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \operatorname{Re}((z_1 + z_3 + (1 - z_1)z_2)e^{-i\alpha}) < \eta \cos \alpha \right\},$$

to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in $\mathfrak{H}_\eta$, and, in addition, $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of the function $F_M(\alpha_1, 1, 1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})$ in the domain $\mathfrak{H}_\eta$.

References

- [1] I. Nyzhnyk, R. Dmytryshyn, T. Antonova, *On branched continued fraction expansions of hypergeometric functions F_M and their ratios*, Modern Math. Methods, 2025, 3, 1-13.

Coefficient Estimates for Bounded Holomorphic Functions

Viktor Savchuk^a, Maryna Savchuk^b

^a*Institute of Mathematics of NAS of Ukraine, Kyiv 01004, Ukraine*

^b*National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv 03057, Ukraine*

(e-mail: savchuk@imath.kiev.ua, ma.savchuk@kpi.ua)

Let f be holomorphic function in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let

$$f(z) = \sum_{k=0}^{\infty} \widehat{f}_k z^k, \quad \widehat{f}_k := \frac{f^{(k)}(0)}{k!},$$

be its Taylor series.

Denote by B the Schur class consisting on all holomorphic functions f in $\mathbb{D}$ with $\sup_{z \in \mathbb{D}} |f(z)| \leq 1$.

This talk is devoted to the following extremal problem: for given n -tuple $\mathbf{k} := (k_1, k_2, \dots, k_n)$, where $k_i < k_j$ for $i < j$, $k_j \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, evaluate

$$S_n(\mathbf{k}) := \sup_{f \in B} \sum_{j=1}^n \left| \widehat{f}_{k_j} \right|,$$

and describe the extremal functions for which supremum is attained.

We evaluate $S_n(\mathbf{k})$ for specified n -tuples in case $n = 2, 3$.

Theorem 1. *Let $\mathbf{k} = (k_1, k_2, k_3)$ be 3-tuple such that $k_3 \geq 2k_2 + 1$. Then*

$$S_3(\mathbf{k}) = \frac{3}{2}.$$

The functions

$$f(z) = e^{i\theta} \frac{c_1 z^{k_1} + c_2 z^{k_2} + z^{k_3}}{1 + \overline{c_1} z^{k_3-k_1} + \overline{c_2} z^{k_3-k_2}},$$

where $c_1, c_2 \in \mathbb{C}$ are such that $|c_1| = |c_2| = 1/2$ and

$$(k_3 - k_1) \arg c_2 - (k_3 - k_2) \arg c_1 \neq \frac{\pi}{2} (2k_3 - k_2 - k_1),$$

and θ is real number, are extremal for $S_3(\mathbf{k})$.

Pitt-Type Inequalities for General Monotone Functions

Niyaz Tokmagambetov

Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
(e-mail: tokmagambetov@math.kz)

We study Pitt-type results for the Fourier transform. A new class of general monotone functions is introduced as a subclass of BV functions, and basic properties are established. It is shown that $GM(\mathbb{R}; \tau, \kappa)$ is a natural generalization of the classical general monotone functions first introduced by Liflyand and Tikhonov in 2008 [1]. Pitt's inequality is proven for functions from this class for the range of weight parameters extending known results.

References

- [1] E. Liflyand and S. Tikhonov, *Extended solution of Boas' conjecture on Fourier transform*, C. R. Math. Acad. Sci. Paris, 2008, 346, 1137–1142.

NON-HARMONIC ANALYSIS**Asymptotic Properties of Bivariate Bernstein-Chlodowsky Polynomials on a Triangle****Aytekin Abdullayeva^a, Aynur Mammadova^b, Lale Aliyeva^c**^{a,b}*Military Institute named after Heydar Aliyev*^c*Institute of Mathematics, Baku, Azerbaijan*

(e-mail: aytekinabdullayeva2@gmail.com, ay.mammadova@yahoo.com, Laleliyeveva703@gmail.com)

This paper introduces and analyzes a sequence of bivariate Bernstein-Chlodowsky polynomials defined on an unbounded triangular domain. The main contribution is the development of approximation results for the operators acting on functions defined on the positive quadrant and satisfying suitable growth conditions. These results significantly extend classical approximation properties of Bernstein-type operators from bounded domains to unbounded triangular domains.

Let f be a continuous real function defined on $[0, \infty)$. In 1932, Bernstein's follower Chlodowsky had constructed the increasing sequences of polynomials

$$B_n(f; x) = \sum_{k=0}^n f\left(\frac{kb_n}{n}\right) C_n^k\left(\frac{x}{b_n}\right) \left(1 - \frac{x}{b_n}\right)^{n-k}, \quad 0 \leq x \leq b_n,$$

that was named as Bernstein-Chlodowsky polynomials some later.

Here

$$\lim_{n \rightarrow \infty} b_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0.$$

Note that, the condition imposed on b_n in the definition of classic Bernstein-Chlodowsky operators doesn't provide continuous convergence of $B_n(f; x)$ polynomials to the function $f(x)$.

We have

$$B_n(f; x) - x^2 = \frac{x(b_n - x)}{n} \text{ for the function } f(t) = t^2.$$

Therefore,

$$\sup_{x \in [0, b_n]} |B_n(t^2; x) - x^2| = \frac{b_n^2}{4n}.$$

Then for convergence to zero as $n \rightarrow \infty$ of the right hand part of this equality does not suffice the fulfillment of condition $\frac{b_n}{n} \rightarrow \infty$.

We'll impose the stronger condition on b_n $\lim_{n \rightarrow \infty} \frac{b_n}{\sqrt{n}} = 0$. Hence it follows that for any natural number r

$$\lim_{n \rightarrow \infty} \frac{b_n^r}{n^{\frac{r}{2}}} = 0.$$

Assume that $\{b_n\}_{n=1}^{\infty}$ is a sequence of positive and increasing numbers such that

$$\lim_{n \rightarrow \infty} b_n = \infty,$$

$$\Delta_{b_n} = \{(x, y) \in R_+^2 : x + y \leq b_n\}.$$

$$R_+^2 = \{(x, y) \in R^2 : x \geq 0, y \geq 0\},$$

It is clear that, for each fixed n , Δ_{b_n} is a closed triangle in the first quadrant of the plane with boundary $x + y = b_n$. Moreover, $\Delta_{b_n} \rightarrow R_+^2$, $n \rightarrow \infty$.

Assume that the function f is defined on R_+^2 and is bounded on the triangle Δ_{b_n} for every fixed n , and satisfies the condition

$$\lim_{n \rightarrow \infty} M_n(f) \frac{b_n^2}{n} = 0$$

Were

$$M_n(f) = \sup_{(t, \tau) \in \Delta_{b_n}} |f(t, \tau)|$$

Let

$$B_n(f; x, y) = \sum_{k=0}^n C_n^k \left(1 - \frac{x+y}{b_n}\right)^{n-k} \sum_{i=0}^k f\left(\frac{k-i}{n}b_n, \frac{i}{n}b_n\right) C_k^i \left(\frac{x}{b_n}\right)^{k-i} \left(\frac{y}{b_n}\right)^i.$$

Theorem. Let f be a function defined on , bounded on the triangle Δ_{b_n} for every fixed n , and satisfies the condition $\lim_{n \rightarrow \infty} M_n(f) \frac{b_n^2}{n} = 0$. Assume that the bivariate function f possesses partial derivatives of all orders up to and including the second order at every fixed point (x, y) of the triangle Δ_{b_n} . Then, for the Bernstein-Chlodowsky polynomial defined on the triangle, the following asymptotic equality holds:

$$B_n(f; x, y) = f(x, y) + \frac{x(b_n - x)}{2n} f_{x^2}''(x, y) +$$

$$+\frac{y(b_n-y)}{2n}f''_{y^2}(x,y)-\frac{xy}{n}f''_{xy}(x,y)+r_n(x,y),$$

where $\lim_{n \rightarrow \infty} r_n(x, y) = 0$.

Note that the asymptotic properties have been studied for the Sass operator as well.

References

- [1] A.D. Gadjiev and N. Ispir, *On a sequence of linear positive operators in weighted spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. 11(19),(1999), 45-55.
- [2] Abdullayeva, A.E., Soylemez, D.: *Approximation of functions by linear positive operators in variable Lebesgue spaces*. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics, 38(4) (2018), 3-7.
- [3] Abdullayeva, A.E., Mammadova, A.N.: *On order of approximation function by generalized Szasz operators and Bernstein-Chlodowsky polynomials*. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., 38 (2013), 3-8.
- [4] Abdullayeva, A.E.: *On Varanovskaya Type Theorem for Generalized Bernstein-Chlodowsky Polynomials*. Asian Res. J. Math., 7(2) (2017), 1-7.
- [5] Mammadova, A.N., Bandaliyev R.A.: *On approximation theorems for two-dimensional Szasz operator*, Baku- Transactions of NAS Azerbaijan, Series of Physical- Technical and Math. Sciences, 37(1), 2017, 53-62.

On the Boundedness of One-Dimensional Hausdorff Operators in Orlicz Spaces

Rovshan Bandaliyev^{a,b}, Aytakin Abdullayeva^a, Dunya Aliyeva^a

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
(e-mail: bandaliyevr@gmail.com)

In this abstract, we establish the necessary condition and sufficient condition on kernel function related to Young function for the boundedness of one-dimensional Hausdorff operator in Orlicz spaces.

It is well known that one-dimensional Hausdorff operators have a deep roots in the study of Fourier analysis, and they have a long history in the study of harmonic analysis. This integral operator is deeply rooted in the study of one-dimensional Fourier analysis and has become an essential part of modern harmonic analysis. In particular, it is closely related to the summability of the classical Fourier series. Let $x \in \mathbb{R}_+ = (0, \infty)$ and let ψ be a fixed kernel defined on $(0, \infty)$ such that $\psi \in L_1^{loc}(\mathbb{R}_+)$. Then the one-dimensional Hausdorff operator is defined by a linear operator

$$H_\psi f(x) = \int_0^\infty \frac{\psi\left(\frac{x}{y}\right)}{y} f(y) dy.$$

Many important operators in analysis are special cases of the Hausdorff operator, which can be obtained by taking suitable choice of ψ . For example, the Hardy operator, the adjoint Hardy operator, the Hardy-Littlewood-Polya(Calderon) operator, Cesaro operator, Hardy-Hilbert operator and so on are particular cases of the Hausdorff operator (see, [1] and [4]).

We recall the definition of Young function. A function $\Phi : [0, \infty) \rightarrow [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{s \rightarrow +0} \Phi(s) = \Phi(0) = 0$ and $\lim_{s \rightarrow \infty} \Phi(s) = \infty$. From the convexity and $\Phi(0) = 0$ it follows that any Young function is non-decreasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. By $\mathcal{R}$ we will denote the set of Young functions such that $0 < \Phi(s) < \infty$ for $s > 0$. If $\Phi \in \mathcal{R}$, then Φ is absolutely continuous on every

closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself. Also the following Lemma holds.

Lemma 1 ([3]). *Let Φ be a Young function. Then, as a convex function, Φ is continuous if and only if Φ is finite on $[0, \infty)$.*

In particular, $\Phi \in C([0, \infty))$ if $\Phi \in \mathcal{R}$. For a Young function Φ let $\Psi(r) = \inf \{s \geq 0 : \Phi(s) > r\}$. It is obvious that, if $\Phi \in \mathcal{R}$, then Ψ is the usual inverse function of Φ . Also, Ψ is non-decreasing and right-continuous function such that $\Psi(0) = 0$. It is easy to see that $\Phi(\Psi(x)) \leq x \leq \Psi(\Phi(x))$ (see, [3]). A Young function Φ satisfies Δ_2 -condition if there exists $M \geq 2$ such that $\Phi(2s) \leq M\Phi(s)$ for all $s > 0$. If $\Phi \in \mathcal{R}$, then $\Phi \in \Delta_2$.

Definition 1. *Let Φ be a Young function. The set*

$$L^\Phi(\mathbb{R}_+) = \left\{ f \in L_1^{loc}(\mathbb{R}_+) : \int_0^\infty \Phi(k|f(x)|) dx < \infty \text{ for some } k > 0 \right\}$$

is called an Orlicz space. $L^\Phi(\mathbb{R}_+)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_0^\infty \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

We have the following theorem.

Theorem 1. *Let Φ be a Young function. Suppose that $\Phi_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing and left-continuous function. Let $\Phi(st) \geq \Phi_1(s)\Phi(t)$ for all $s, t \geq 0$ and let*

$$A = \int_0^\infty \frac{|\psi(y)|}{y} \Phi_1^{-1}(y) dy < \infty.$$

Then

$$\|H_\psi f\|_{L^\Phi} \leq A \|f\|_{L^\Phi}. \quad (1)$$

In the proof of Theorem 1, the boundedness of the dilation operator in the Orlicz space is used significantly. For detail, see [2].

Now about the converse of Theorem 1.

Theorem 2. *Suppose that ψ is a fixed positive function on $\mathbb{R}_+$. Let Φ be a Young function and suppose that $\Phi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function. Let $\Phi^{-1}(st) \geq \Phi_2(s)\Phi^{-1}(t)$ for all $s, t \geq 0$. Suppose that the inequality $\|H_\psi f\|_{L^\Phi} \leq C \|f\|_{L^\Phi}$ holds.*

Then $LB \leq C$, where

$$B = \int_0^{\infty} \frac{\psi(y)}{y} \Phi_2(y) dy$$

$$\text{and } L = \lim_{\varepsilon \rightarrow +0} \frac{\|\Phi^{-1}(x^{-1-\varepsilon})\|_{L^{\Phi}(\frac{1}{\varepsilon}, \infty)}}{\|\Phi^{-1}(x^{-1-\varepsilon})\|_{L^{\Phi}(1, \infty)}}.$$

Corollary 1. Suppose that ψ is a fixed positive function on $\mathbb{R}_+$. Let $\Phi \in \mathcal{R}$ be a Young function satisfying all conditions of Theorem 1 and Theorem 2. Suppose that there exists $K > 0$ such that $\Phi_1^{-1}(y) \leq K \Phi_2(y)$ for almost everywhere $y \geq 0$. Then the inequality (1) holds if and only if $B < \infty$.

Corollary 2. Suppose that ψ is a fixed positive function on $\mathbb{R}_+$. Let $1 \leq p < \infty$ and let $\Phi(t) = \Phi_1(t) = \Phi_2(t) = t^p$.

Then the inequality (1) holds if and only if

$$\int_0^{\infty} \frac{\psi(y)}{y} y^{\frac{1}{p}} dy < \infty.$$

References

- [1] R.A. Bandaliyev, P. Górká, *Hausdorff operator in Lebesgue spaces*, Mathematical Inequalities & Applications, 2019, 22(2), 657-676.
- [2] O. Blasco, A. Osançhol, *Notes on bilinear multipliers on Orlicz spaces*, Mathematische Nachrichten, 2019, 292(12), 2522-2536.
- [3] L. Diening, P. Harjulehto, P. Hasto, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.
- [4] E. Liflyand, A. Mirotn, *Hausdorff operators: problems and solutions*, Journal of Mathematical Sciences, 2024, 280(6), 1065-1114.

On Kolmogorov-Riesz Compactness Theorem in Quasi-Grand Lebesgue Space

Rovshan Bandaliyev^{a,b}, Vugar Khalilov^b, Khalida Nasirova^a

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
(e-mail: bandaliyevr@gmail.com)

In this abstract, we give an analog of the Kolmogorov-Riesz compactness theorem in quasi-grand Lebesgue space $L_p)(0, 1)$, $0 < p \leq 1$. Also, the authors study some problems of separability in $L_p)(0, 1)$.

Grand Lebesgue spaces have become a natural tool in various applications both in problems of analysis, for example, in the study of the integrability of the Jacobian, see [7], and in the theory of differential equations, for example, in the problem of inverting the p -harmonic operator, see [4]. It is well known that the grand Lebesgue space $L_p)(\Omega)$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$, was introduced in [7] and defined by the norm

$$\|f\|_{L_p)(\Omega)} = \sup_{0 < \varepsilon < p-1} \left(\frac{\varepsilon}{|\Omega|} \int_{\Omega} |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

The theory of grand Lebesgue spaces is developed by many mathematicians. In particular, the study of grand Lebesgue spaces has attracted resurgent attentions in recent years. We refer to [3], [6], [9] for some recent work in this vein. But, quasi-grand Lebesgue space $L_p)(0, 1)$, $0 < p \leq 1$, was introduced in [1] and defined by the norm

$$\|f\|_{L_p)(0,1)} = \sup_{0 < \varepsilon < \frac{p}{2}} \left(\varepsilon \int_0^1 |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

Next, the extension of the definition of grand(quasi-grand) Lebesgue space to the rearrangement invariant Banach function was given in [2]. It is well known that the grand(quasi-grand) Lebesgue spaces are not separable.

Let us note the following well-known useful properties of the quasi-grand Lebesgue space $L_p)(0, 1)$.

Theorem 1.[1] *Let $0 < p \leq 1$. The space $L_p(0, 1)$ is a quasi-Banach function space.*

Lemma 1.[1] *Let $0 < p \leq 1$ and let $0 < \varepsilon < \frac{p}{2}$. Then the following statements hold*

- (i) $L_p(0, 1) \hookrightarrow L_p(0, 1) \hookrightarrow L_{p-\varepsilon}(0, 1)$ for any $0 < \varepsilon < \frac{p}{2}$;
- (ii) if $p_1 \leq p_2$, then $L_{p_2}(0, 1) \hookrightarrow L_{p_1}(0, 1)$.

It is well known that the closure of compactly supported smooth functions $C_0^\infty(0, 1)$ with respect to the grand(quasi-grand) Lebesgue norm does not coincide with the grand(quasi-grand) Lebesgue space. Let us consider a special notation for it, namely, will stand for the closure of $C_0^\infty(0, 1)$ with respect to the quasi-grand Lebesgue norm. In the following proposition, it is shown that this closure can be characterized as

$$L_p^0(0, 1) = \left\{ f \in L_p(0, 1) : \lim_{\varepsilon \rightarrow 0+} \varepsilon \int_0^1 |f(x)|^{p-\varepsilon} dx = 0 \right\}.$$

Theorem 2. *Let $0 < p \leq 1$. The space of infinitely differentiable functions with compact support $C_0^\infty(0, 1)$ is not dense in $L_p(0, 1)$. Moreover, $\overline{C_0^\infty(0, 1)} = L_p^0(0, 1)$.*

Now we give an analog of the Kolmogorov-Riesz compactness theorem in quasi-grand Lebesgue space. About the history of Kolmogorov-Riesz compactness Theorem, we refer, for example, to the paper [5].

We have the following main theorem.

Theorem 3. *Let $0 < p \leq 1$. A subset $\mathcal{F} \subset L_p^0(0, 1)$ is totally bounded if and only if*

- (i) $\mathcal{F}$ is uniformly bounded, i.e. $\sup_{f \in \mathcal{F}} \|f\|_{L_p(0, 1)} < \infty$.
- (ii) $\mathcal{F}$ is uniformly equicontinuous, i.e. $\sup_{f \in \mathcal{F}} \|f(\cdot + h) - f\|_{L_p(0, 1)} \rightarrow 0$ for $h \rightarrow 0$.

Remark 1. We observe that an analog of the Kolmogorov-Riesz compactness theorem in grand Lebesgue space $L_p(\Omega)$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$, was proved in [10]. Also, the compactness of families of continuous functions in grand Lebesgue spaces was studied in [8].

References

- [1] R.A. Bandaliyev, K.H. Safarova, *On Hardy type inequalities in grand Lebesgue spaces $L_p(0, 1)$ for $0 < p \leq 1$* , Linear and Multilinear Algebra, 2022, 70(21), 6053-6066.
- [2] M.R. Formica, E. Ostrovsky, L. Sirota, *On grand quasi Lebesgue space*, Journal of Mathematical Analysis and Applications, 2021, 504(1), art.125369, 1-21.
- [3] G. Di Fratta, A. Fiorenza, *A direct approach to the duality of grand and small Lebesgue spaces*, Nonlinear Analysis: Theory, Methods & Applications, 2009, 70(7), 2582-2592.
- [4] L. Greco, T. Iwaniec, C. Sbordone, *Inverting the p -harmonic operator*, Manuscripta Mathematica, 1997, 92(1), 249-258.
- [5] H. Hanche-Olsen, H. Holden, *The Kolmogorov–Riesz compactness theorem*, Expositiones Mathematicae, 2010, 28(4), 385–394.
- [6] M.I. Ismailov, Ch.M. Hasanova, H.Z. Ashirova, *On some properties of grand sequence spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.Mathematics, 2025, 45(4), 88-96.
- [7] T. Iwaniec, C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Archive for Rational Mechanics and Analysis, 1992, 119(2), 129-143.
- [8] R. Katea, Y. Kaya, *On the compactness of families of continuous functions in grand Lebesgue spaces*, AIMS Mathematics, 2026, 11(5), 13647–13659.
- [9] V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-standard Function Spaces*, Vol. 2: Variable exponent Hölder, Morrey-Campanato and Grand Spaces. Birkhauser, 2016.
- [10] H. Rafeiro, A. Vargas, *On the compactness in grand spaces*, Georgian Mathematical Journal, 2015, 22(1), 141–152.

On the Integrability of the Boundary Values of Harmonically Conjugate Functions

Aynur Huseynli

Baku State University, Baku, Azerbaijan
(e-mail: aynurhuseynli88@gmail.com)

For a measurable complex-valued function f on R^d we set

$$[f(x)]_{\delta,\lambda} = [f(x)]^{\delta,\lambda} = f(x) \quad \text{for } \delta \leq |f(x)| \leq \lambda,$$

$$[f(x)]_{\delta,\lambda} = [f(x)]^{\delta,\lambda} = 0 \quad \text{for } |f(x)| < \delta,$$

$$[f(x)]_{\delta,\lambda} = \lambda \cdot \operatorname{sgn} f(x), \quad [f(x)]^{\delta,\lambda} = 0 \quad \text{for } |f(x)| > \lambda.$$

Definition 1. If the finite limit

$$\lim_{\substack{\delta \rightarrow 0+ \\ \lambda \rightarrow +\infty}} \int_{R^d} [f(x)]_{\delta,\lambda} dx \left(\lim_{\substack{\delta \rightarrow 0+ \\ \lambda \rightarrow +\infty}} \int_{R^d} [f(x)]^{\delta,\lambda} dx \right)$$

exists, then f is said to be Q -integrable (Q' -integrable, respectively) on R^d , and this fact is written as $f \in Q(R^d)$ ($f \in Q'(R^d)$, respectively). The values of this limit is referred to as the Q -integral (Q' -integral, respectively) of the function f and is denoted by

$$(Q) \int_{R^d} f(x) dx \quad \left((Q') \int_{R^d} f(x) dx \right).$$

Note that, as in the case of an interval, the Q and Q' -integrals of the functions measurable on R^d do not satisfy additivity property ([1], [2], [3]). If one adds the conditions

$$m \{ x \in R^d : |f(x)| > \delta \} = o(1/\delta), \quad \delta \rightarrow 0+, \quad (1)$$

$$m \{ x \in R^d : |f(x)| > \lambda \} = o(1/\lambda), \quad \lambda \rightarrow +\infty, \quad (2)$$

to the definition of Q -integrability (Q' -integrability), then the Q -integral and the Q' -integral coincide ($Q(R^d) = Q'(R^d)$) and these integrals become additive.

Definition 2. If $f \in Q'(R^d)$ (or $f \in Q(R^d)$) and the conditions (1) and (2) hold, then f is said to be A -integrable on R^d , and this fact is written as $f \in A(R^d)$. The limit

$$\lim_{\substack{\delta \rightarrow 0+ \\ \lambda \rightarrow +\infty}} \int_{R^d} [f(x)]_{\delta, \lambda} dx \quad \left(\text{or} \quad \lim_{\substack{\delta \rightarrow 0+ \\ \lambda \rightarrow +\infty}} \int_{R^d} [f(x)]^{\delta, \lambda} dx \right)$$

is denoted in this case by

$$(A) \int_{R^d} f(x) dx.$$

Definition 3. If the functions $u_0(x, x_0), u_1(x, x_0), \dots, u_d(x, x_0)$ defined in the upper half-space $R_+^{d+1} = \{ (x, x_0) \in R^{d+1} : x \in R^d, x_0 > 0 \}$ satisfy the following generalized Cauchy-Riemann equations:

$$\begin{cases} \sum_{j=0}^d \frac{\partial u_j}{\partial x_j} = 0, \\ \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \quad j, k = 0, 1, \dots, d, \end{cases}$$

then they are called conjugate harmonic functions in the upper half-space.

Theorem. If the boundary values of a harmonic function $u(x, x_0)$ on the upper half-space R_+^{d+1} are Lebesgue integrable, then the harmonically conjugate functions $u_j(x, x_0)$, $j = 1, \dots, d$ are the Poisson A -integrals of their boundary values, that is

$$u_j(x, x_0) = (A) \int_{R^d} P_{x_0}(x-t) f_j(t) dt, \quad j = 1, \dots, d,$$

where $P_{x_0}(x) = \gamma(d) \cdot \frac{x_0}{(|x|^2 + x_0^2)^{\frac{d+1}{2}}}$ is the Poisson kernel and $f_j(x)$, $j = 1, \dots, d$ are the boundary values of the functions $u_j(x, x_0)$.

References

- [1] Aleksandrov, A.B.: A -integrability of the boundary values of harmonic functions. Math. Notes Acad. Sci. USSR 30(1), 515-523 (1981).
- [2] Aliev, R.A.: On the properties of Q and Q' -integrals of the function measurable on the real axis. Proc. Inst. Math. Nat. Acad. Sci. Azerb. 41(1), 56-62 (2015).
- [3] E.C.Titchmarsh, On conjugate functions, Proceedings of the London Mathematical Society, 1929, 9, 49–80.

Gegenbauer Fractional Integral Operator and its Commutators on Gegenbauer-Orlicz Spaces

Elman Ibrahimov^a, Aynura Binnatova^b, Saadat Jafarova^c

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Baku Slavic University, Baku, Azerbaijan*

^c*Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan*

(e-mail: elmanibrahimov@yahoo.com, aynura.binnatova@gmail.com,
sada-jafarova@rambler.ru)

In this abstract, we give criteria for the strong and weak boundedness of the G -fractional integral J_G^α on G -Orlicz spaces. As an application of these results, we consider the boundedness of the commutators of the G -fractional integral $J_G^{b,\alpha}$ on Orlicz spaces when b belongs to the G -BMO and G -Lipschitz spaces, respectively.

1. Preliminaries: on Young functions and G -Orlicz spaces

Let's recall the definition of Young functions.

Definition 1.1. A function $\Phi : [0, \infty) \rightarrow [0, \infty]$ is called a Young function if Φ is convex and left-continuous, $\lim_{r \rightarrow +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \rightarrow \infty} \Phi(r) = \infty$. Due to the convexity and $\Phi(0) = 0$, it follows that any Young function is increasing.

Definition 1.2. (G -Orlicz space) For a Young function Φ , the set

$$L_\Phi(\mathbb{R}_+, G) = \left\{ f \in L_1^{loc}(\mathbb{R}_+, G) : \int_{\mathbb{R}_+} \Phi(k|f(chx)|) d\mu_\lambda(x) < \infty \text{ for some } k > 0 \right\}$$

is called Gegenbauer-Orlicz space (G-Orlicz space).

Definition 1.3. The weak G-Orlicz space

$$WL_\Phi(\mathbb{R}_+, G) = \{f \in L_1^{loc}(\mathbb{R}_+, G) : \|f\|_{WL_\Phi} < \infty\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf \left\{ \nu > 0 : \sup_{t>0} \Phi(t) m \left(\frac{f}{\nu}, t \right) \leq 1 \right\}.$$

2. G-fractional integral in G-Orlicz spaces

G-fractional integral J_G^α is defined as follows:

$$J_G^\alpha f(chx) = \int_0^\infty \frac{A_{chy}^\lambda f(chx)}{(sh\frac{y}{2})^{\gamma-\alpha}} d\mu_\lambda(y), \quad 0 < \alpha < \gamma. \quad (8)$$

Theorem 2.1. Let $0 < \alpha < \gamma$ and $\Phi, \Psi \in \mathcal{Y}$. If the inequality

$$\int_r^\infty \Phi^{-1}(t^{-\gamma}) t^{\alpha-1} dt \leq c \Psi^{-1}(r^{-\gamma}) \quad (9)$$

holds for all $r > 0$, where $c > 0$ does not depend on r , then the condition

$$r^\alpha \Phi^{-1}(r^{-\gamma}) \leq c \Psi^{-1}(r^{-\gamma})$$

is necessary and sufficient for the boundedness of J_G^α from $L_\Phi(\mathbb{R}_+, G)$ to $WL_\Psi(\mathbb{R}_+, G)$.

3. Commutators of G-fractional integral in G-Orlicz spaces

We recall the definition of the G-BMO space which is defined as follows:

$$BMO_G(\mathbb{R}_+) = \{f \in L_1^{loc}(\mathbb{R}_+, G) \setminus \text{const} : \|f\|_G^* < \infty\},$$

where

$$\|f\|_G^* = \sup_{x, r \in \mathbb{R}_+} \frac{1}{|H_r|_\lambda} \int_{H_r} |A_{chy}^\lambda f(chx) - b_{H_r}(chx)| d\mu_\lambda(y)$$

is the norm, and

$$f_{H_r}(chx) = \frac{1}{|H_r|_\lambda} \int_{H_r} A_{chy}^\lambda f(chx) d\mu_\lambda(y).$$

Theorem 3.1. Let $0 < \alpha < \gamma$, $b \in BMO_G(\mathbb{R}_+)$. Let $\Phi \in \Delta_2 \cap \nabla_2$. If the condition

$$\int_r^\infty \left(1 + \frac{t^{-\gamma}}{r^{-\gamma}}\right) \Phi^{-1}(t^{-\gamma}) t^{\alpha-1} dt \lesssim r^\alpha \Phi^{-1}(r^{-\gamma}) \quad (10)$$

holds for all $r > 0$, then the condition (2.2) is necessary and sufficient for the boundedness of $|J_G^{b,\alpha}|$ from $L_\Phi(\mathbb{R}_+, G)$ to $L_\Psi(\mathbb{R}_+, G)$.

Theorem 3.2. Let $0 < \alpha < \gamma$, $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ and the condition (3.1) holds. If $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\alpha/\gamma}$, then the condition $b \in BMO_G(\mathbb{R}_+)$ is necessary and sufficient for the boundedness of $J_G^{b,\alpha}$ from $L_\Phi(\mathbb{R}_+, G)$ to $L_\Psi(\mathbb{R}_+, G)$.

4. Characterization of Gegenbauer-Lipschitz spaces via commutators of the G-fractional integral

Definition 4.1. Let $0 < \beta < 1$. We say a function b belongs to the G-Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}_+, G)$ if there exists a constant C such that, the following inequality

$$|A_{chy}^\lambda b(chx) - b_{H_r}(chx)| \leq C(chy - 1)^\beta,$$

where $f_{H_r}(chx) = \frac{1}{|H_r|_\lambda} \int_{H_r} A_{chy}^\lambda f(chx) d\mu_\lambda(y)$, holds for all $x, y \in \mathbb{R}_+$.

The smallest such C is called the $\dot{\Lambda}_\beta(\mathbb{R}_+, G)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}_+, G)}$. In order to prove the theorems, we need auxiliary results.

Theorem 4.2. Let $0 < \beta < 1$, $0 < \alpha < \gamma$, $0 < \alpha + \beta < \gamma$, $b \in L_1^{loc}(\mathbb{R}_+, G)$ and $\Phi, \Psi \in \mathcal{Y}$. Let $\Phi \in \nabla_2$ and the condition

$$\int_t^\infty \Phi^{-1}(r^{-\gamma}) r^{\alpha+\beta-1} dr \lesssim t^{\alpha+\beta} \Phi^{-1}(t^{-\gamma})$$

hold for every $t > 0$. If

$$\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-(\alpha+\beta)/\gamma},$$

then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$ is necessary and sufficient for the boundedness $|J_G^{b,\alpha}|$ from $L_\Phi(\mathbb{R}_+, G)$ to $L_\Psi(\mathbb{R}_+, G)$.

Description of the Generalized Besov Spaces in Terms of Mean Oscillation

Rahim Rzaev^{a,b}, Lale Aliyeva^c, Surayye Isayeva^b

^a*Baku State University, Baku, Azerbaijan*

^b*Azerbaijan State Pedagogical University, Baku, Azerbaijan*

^c*Institute of Mathematics, Baku, Azerbaijan*

(e-mail: rrzaev@rambler.ru, lalaliyeva703@gmail.com,
surayye.necefova@gmail.com)

Let R^n be the n -dimensional Euclidean space of points $x = (x_1, x_2, \dots, x_n)$, $B(a, r) := \{x \in R^n : |x - a| \leq r\}$ is a closed ball in R^n of radius $r > 0$ with the center at the point $a \in R^n$, N is the set of all natural numbers, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $x^\nu = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdot \dots \cdot x_n^{\nu_n}$, $|\nu| = \nu_1 + \nu_2 + \dots + \nu_n$, where $\nu_1, \nu_2, \dots, \nu_n$ are nonnegative integers. The set of all functions whose p -th degree of modulus is locally summable in R^n is denoted by $L_{loc}^p(R^n)$ ($1 \leq p < \infty$), and the set of all locally bounded functions is denoted by $L_{loc}^\infty(R^n)$.

Let $f \in L_{loc}^1(R^n)$, $k \in N \cup \{0\}$. Consider a polynomial (see [1], [2])

$$P_{k,B(a,r)}f(x) := \sum_{|\nu| \leq k} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} f(t) \varphi_\nu \left(\frac{t-a}{r} \right) dt \right) \varphi_\nu \left(\frac{x-a}{r} \right),$$

where $|B(a,r)|$ denotes the volume of the ball $B(a,r)$ and $\{\varphi_\nu\}$, $|\nu| \leq k$, the orthonormalized system obtained as a result of the orthogonalization process with respect to the scalar product

$$(f, g) := \frac{1}{|B(0,1)|} \int_{B(0,1)} f(t)g(t)dt$$

to the system of power functions $\{x^\nu\}$, $|\nu| \leq k$, arranged in a partially lexicographic order [3].

$P_{k,B(a,r)}f$ – polynomial of degree not higher than k . The set of all polynomials R^n to the degree not higher k is denoted by P_k . Thus, $P_{k,B(a,r)}f \in P_k$.

Let $f \in L_{loc}^p(R^n)$, $1 \leq p \leq \infty$, $k \in N$. Let's consider the function

$$M_f^k(x; r)_p := \inf_{\pi \in P_{k-1}} |B(x, r)|^{-\frac{1}{p}} \|f - \pi\|_{L^p(B(x, r))} \quad (r > 0, x \in R^n).$$

It is not difficult to verify that $M_f^k(x; r)_p \asymp \Omega_k(f, B(x, r))_p$, ($x \in R^n, r > 0$), where

$$\begin{aligned}\Omega_k(f, B(x, r))_p &:= |B(x, r)|^{-\frac{1}{p}} O_k(f, B(a, r))_p, \\ O_k(f, B(x, r))_p &:= \|f - P_{k-1, B(a, r)} f\|_{L^p(B(a, r))}.\end{aligned}$$

The quantity $\Omega_k(f, B(x, r))_p$ is called the mean oscillation of the k -th order of the function f in the ball $B(a, r)$ in the metric L^p .

Let $1 \leq p, q \leq \infty$. Let's introduce the function

$$M_f^k(r)_{pq} := \begin{cases} \|M_f^k(\cdot; r)_p\|_{L^q(R^n)}, & \text{for } 1 \leq q < \infty, \\ \sup_{x \in R^n} M_f^k(x; r)_p, & \text{for } q = \infty. \end{cases}$$

The modulus of continuity of the k -th order ($k \in N$) of a function f in a metric L^p ($1 \leq p \leq \infty$) is determined by the equality of

$$\omega_f^k(r)_p := \sup \left\{ \|\Delta_h^k f\|_{L^p(R^n)} : |h| \leq r \right\}, (r > 0),$$

where $\Delta_h^1 f(x) := f(x+h) - f(x)$, $\Delta_h^k f = \Delta_h^1(\Delta_h^{k-1} f)$.

Let Φ be a class of all positive monotonically increasing on $(0, +\infty)$ functions $\varphi(t)$ such that $\varphi(+0) = 0$. The set of all functions $\varphi \in \Phi$ for which $\frac{\varphi(t)}{t^\nu}$ almost decreases (see [3]) is denoted by Φ_k .

Let $k \in N$, $\varphi \in \Phi_k$, $1 \leq p, q \leq \infty$. By $B_{p,q}^{k,\varphi}$ we denote the totality of all the functions $f \in L^p(R^n)$ (for $p \leq \infty$ it is assumed that f is equivalent to a continuous function) for which the following semi-norm is finite

$$|f|_{B_{p,q}^{k,\varphi}} := \begin{cases} \left(\int_0^\infty \left(\frac{\omega_f^k(t)_p}{\varphi(t)} \right)^q \frac{dt}{t} \right)^{1/q}, & \text{if } 1 \leq q < \infty, \\ \sup \left\{ \frac{\omega_f^k(t)_p}{\varphi(t)} : t > 0 \right\}, & \text{if } q = \infty. \end{cases}$$

In the space $B_{p,q}^{k,\varphi}$ we introduce the norm by the equality

$$\|f\|_{B_{p,q}^{k,\varphi}} = \|f\|_{L^p(R^n)} + |f|_{B_{p,q}^{k,\varphi}}.$$

Let $\varphi \in \Phi_k$, $k \in N$, or $\varphi(t) \equiv 1$, $1 \leq p, q, \theta \leq \infty$. By $MO_{p,q,\theta}^{k,\varphi}$ we denote the class of all functions $f \in L_{loc}^p(R^n)$ such that $\|f\|_{MO_{p,q,\theta}^{k,\varphi}} < +\infty$, where

$$\|f\|_{MO_{p,q,\theta}^{k,\varphi}} := \begin{cases} \left(\int_0^\infty \left(\frac{M_f^k(t)_{pq}}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{1/q}, & \text{if } 1 \leq \theta < \infty, \\ \sup \left\{ \frac{M_f^k(t)_{pq}}{\varphi(t)} : t > 0 \right\}, & \text{if } \theta = \infty. \end{cases}$$

The class $MO_{p,q,\theta}^{k,\varphi}$ is regarded as a subset in the quotient space $f \in L_{loc}^p(R^n) / P_{k-1}$ and $MO_{p,q,\theta}^{k,f}$ is a Banach space with the norm $\|f\|_{MO_{p,q,\theta}^{k,\varphi}}$.

Theorem 1. Let $k \in N$, $1 \leq q \leq p \leq \infty$, $q < \infty$, $1 \leq \theta \leq \infty$, $\varphi \in \Phi_k$ and

$$\int_0^\delta \frac{\varphi(t)}{t} dt = O(\varphi(\delta)) \quad (\delta > 0).$$

Then $B_{p,\theta}^{k,\varphi} = MO_{q,p,\theta}^{k,\varphi} \cap L^p(R^n)$ and

$$\exists C_1 > 0 \quad \exists C_2 > 0 \quad \forall f : C_1 \|f\|_{B_{p,\theta}^{k,\varphi}} \leq \|f\|_{MO_{q,p,\theta}^{k,\varphi}} + \|f\|_{L^p(R^n)} \leq C_2 \|f\|_{B_{p,\theta}^{k,\varphi}}.$$

Theorem 2. Suppose that $k \in N$, $1 \leq \theta \leq \infty$, $\varphi \in \Phi_k$. Then $B_{\infty,\theta}^{k,\varphi} = MO_{\infty,\infty,\theta}^{k,\varphi} \cap L^\infty(R^n)$ and

$$\exists C_1 > 0 \quad \exists C_2 > 0 \quad \forall f : C_1 \|f\|_{B_{\infty,\theta}^{k,\varphi}} \leq \|f\|_{MO_{\infty,\infty,\theta}^{k,\varphi}} + \|f\|_{L^\infty(R^n)} \leq C_2 \|f\|_{B_{\infty,\theta}^{k,\varphi}}.$$

References

- [1] DeVore R., Sharpley R. Maximal functions measuring smoothness. Memoir. Amer. Math. Soc., 1984, 47(293), 1-115.
- [2] Rzaev R.M. A multidimensional singular integral operator in spaces defined by conditions on the k -th order mean oscillation. Dokl. Akad. Nauk (Russia)., 1997, 356(5), 602-604. (in Russian).
- [3] Rzaev R.M. Properties of singular integrals in terms of maximal functions measuring smoothness. Eurasian Math. J., 2013, 4(3), 107-119.

PROBLEMS OF MECHANICS**Simulation of Stationary Annular Flow of a Gas-Liquid Mixture in a Pipe, Considering Radial Viscosity Variation of the Gas Cushion****Elkhan Abbasov^a, Nurlana Agayeva^b, Aynur Darishova^c***^aInstitute of Mathematics, Baku, Azerbaijan**^bScientific Research and Design Institute Neftegaz of the State Oil Company of the Republic of Azerbaijan, Baku, Azerbaijan**^cAzerbaijan University of Architecture and Construction, Baku, Azerbaijan
(e-mail: n.agayeva1975@gmail.com, aelhan@mail.ru, aynur.darishova@azmiu.edu.az)*

This study presents a mathematical model of steady axisymmetric annular flow of a gas–liquid mixture in a cylindrical pipe, taking into account the radial variation of the dynamic viscosity within the gas cushion. Unlike most existing models, in which the viscosity of the gas phase is assumed to be constant, the proposed approach considers its linear variation along the radial direction. Differential equations governing the motion of both the gas and liquid phases are derived, and the corresponding boundary conditions are formulated at the pipe axis, the gas–liquid interface, and the inner wall of the pipe. It is demonstrated that the incorporation of variable viscosity significantly affects the velocity profile within the gas cushion, the interfacial shear stress, and the hydraulic pressure losses. The proposed mathematical model can be employed for predicting gas–liquid flow characteristics in oil and gas wells, long-distance pipelines, and thermal power engineering equipment.

Mathematical Formulation of the Problem and Flow Model

Consider a steady, axisymmetric annular flow of a gas-liquid mixture in a cylindrical pipe of radius R . In the present model, it is assumed that, under the combined influence of centrifugal and hydrodynamic forces, the gas phase accumulates near the pipe wall, forming an annular gas cushion of thickness $R - R_0$, where R_0 denotes the radius of the gas-liquid interface. Accordingly, the inner region of the pipe,

$0 \leq r \leq R_0$, is occupied by the liquid phase, whereas the annular region $R_0 \leq r \leq R$, contains the gas cushion. The flow of both phases is assumed to

be steady, axisymmetric, incompressible, and fully developed. Consequently, the axial velocity depends solely on the radial coordinate,

$$u = u(r).$$

For the gas region, the steady-state axisymmetric Navier–Stokes equation can be written as

$$-\frac{\Delta P}{L} = \frac{1}{r} \frac{d}{dr} \left(r \mu(r) \frac{du_1}{dr} \right), \quad (1)$$

where ΔP is the pressure drop over a pipe section of length L ; $u_1(r)$ is the axial gas velocity; $\mu(r)$ is the radially varying dynamic viscosity of the gas cushion.

Expanding the derivative in Eq. (1) yields

$$-\frac{\Delta P}{L} = \mu(r) \frac{d^2 u_1}{dr^2} + \left(\frac{\mu(r)}{r} + \frac{d\mu}{dr} \right) \frac{du_1}{dr}. \quad (2)$$

Unlike the classical approach, where the gas viscosity is assumed constant throughout the flow domain, the present model assumes that the dynamic viscosity varies linearly along the radial direction according to

$$\mu(r) = \mu_l - \frac{r - R_0}{R - R_0} (\mu_l - \mu_g), \quad (3)$$

where μ_l is the dynamic viscosity of the liquid, μ_g is the dynamic viscosity of the gas, R_0 is the radius of the gas-liquid interface, R is the inner radius of the pipe.

Differentiating Eq. (3) with respect to the radial coordinate gives

$$\frac{d\mu}{dr} = -\frac{\mu_l - \mu_g}{R - R_0}. \quad (4)$$

Substituting Eqs. (3) and (4) into Eq. (2) results in the governing differential equation describing the motion of the gas phase in a viscosity-stratified gas cushion,

$$\begin{aligned} & \left[\mu_j - \frac{r - R_0}{R - R_0} (\mu_j - \mu_g) \right] \frac{d^2 u_1}{dr^2} + \\ & + \left[\frac{1}{r} \left(\mu_j - \frac{r - R_0}{R - R_0} (\mu_j - \mu_g) \right) - \frac{\mu_j - \mu_g}{R - R_0} \right] \frac{du_1}{dr} = -\frac{\Delta P}{L} \end{aligned} \quad (5)$$

Equation (5) represents the principal governing equation of the proposed model and characterizes the flow of the gas phase with spatially varying viscosity.

For the liquid central region, the viscosity is considered constant: $\mu = \mu_l = \text{const}$.

Under this assumption, the governing equation for the fully developed axisymmetric flow of the liquid phase reduces to the classical form

$$\mu_l \frac{1}{r} \frac{d}{dr} \left(r \frac{du_2}{dr} \right) = -\frac{\Delta P}{L}, \quad (6)$$

where $u_2(r)$ denotes the axial velocity of the liquid phase.

Integrating Eq. (6) with respect to the radial coordinate gives

$$\frac{du_2}{dr} = -\frac{\Delta P}{2\mu_l L} r. \quad (7)$$

A second integration yields the parabolic velocity distribution

$$u_2(r) = C_1 - \frac{\Delta P}{4\mu_l L} r^2, \quad (8)$$

where C_1 is an integration constant to be determined from the boundary conditions.

Since the flow is axisymmetric, the liquid velocity must remain finite at the pipe axis. Therefore, the regularity condition is satisfied,

$$u_2(0) < \infty. \quad (9)$$

At the interface separating the gas and liquid phases ($r = R_0$), both the velocity field and the tangential stress must be continuous.

Accordingly, the continuity of velocity requires

$$u_1(R_0) = u_2(R_0), \quad (10)$$

$$\mu(R_0) \frac{du_1}{dr} \Big|_{r=R_0} = \mu_l \frac{du_2}{dr} \Big|_{r=R_0}. \quad (11)$$

Equation (11) represents the physical condition of equality of the interfacial shear stresses, $t_g = t_l$.

After solving equation (5), taking into account conditions (10)-(11), the velocity profile of the gas cushion is determined and the mass flow rate of each phase of the mixture is determined separately.

Dynamics of Gas-Liquid Mixture Flow in a Pipe, Taking Into Account Phase Transitions and Heat Exchange With the Environment

Elkhan Abbasov^a, Aynur Darishova^b

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
(e-mail: aelhan@mail.ru, aynur.darishova@azmiu.edu.az)

Gas-liquid mixtures are widely used in various industries, including oil and gas, chemical and thermal power.

The transport of multiphase media is accompanied by complex hydrodynamic, thermal, and mass-transfer phenomena. Of particular interest is the flow dynamics of gas-liquid mixtures in pipelines, including phase transitions and heat exchange with the environment, as these processes determine the operational characteristics of pipeline systems.

As a gas-liquid mixture moves along a pipeline, its pressure and temperature continuously change. Due to interactions with the surrounding environment, the mixture can either cool or heat up, which changes its physical properties and phase composition. These phenomena significantly affect phase distribution, component velocities, hydrodynamic resistance, and the intensity of heat transfer in the system.

Therefore, modeling the dynamics of gas-liquid mixture movement in a pipe, taking into account phase transitions and heat exchange with the environment, has both important practical and scientific significance. Advances in computing technology and numerical methods have led to the development of more sophisticated mathematical models of multiphase flows. The use of computational fluid dynamics methods allows us to simulate unsteady flow regimes, take into account the actual properties of working fluids, and analyze the impact of external conditions on flow parameters.

However, modeling gas-liquid flows taking into account phase transitions and heat transfer remains a complex scientific task requiring further improvement of mathematical models and calculation algorithms.

Despite the numerous studies in this area, there remains a need to develop simplified, adequate, but physically sound models capable of taking into account the main effects of heat exchange and phase transformations.

This paper proposes a mathematical model of gas-liquid mixture flow in a pipeline, taking into account phase transitions, heat exchange with the environment, and changes in the rheological properties of the mixture.

The equations for momentum and mass conservation, ignoring frictional forces between the phases, will have the form [1,2]:

$$-\frac{\partial P_i}{\partial x} = \frac{\partial Q_i}{\partial t} + 2aQ_i + V_{i0} \frac{\partial Q_i}{\partial x} - \frac{1}{c_i^2} \frac{\partial P_i}{\partial t} = \frac{\partial Q_i}{\partial x} - \frac{D_i}{c_i^2} \frac{\partial^2 P_i}{\partial x^2} - \rho_i k \frac{\partial P}{\partial t} \quad (i = 1, 2) \quad (1)$$

where $i = 1$ refers to the parameters of the gas phase, $i = 2$ to the parameters of the liquid phase.

From equation (1), taking into account the boundary conditions, the heat influx equation, the law of change in gas density and Henry's law, after linearization we obtain: for the gas phase

$$-\frac{\partial P_i}{\partial x} = \frac{\partial Q_i}{\partial t} + 2aQ_i + V_{i0} \frac{\partial Q_i}{\partial x} - \frac{1}{c_i^2} \frac{\partial P_i}{\partial t} = \frac{\partial Q_i}{\partial x} - \frac{D_i}{c_i^2} \frac{\partial^2 P_i}{\partial x^2} - \rho_i k \frac{\partial P}{\partial t} \quad (i = 1, 2) \quad (2)$$

where $i = 1$ refers to the parameters of the gas phase, $i = 2$ to the parameters of the liquid phase.

From equation (1), taking into account the boundary conditions, the heat influx equation, the law of change in gas density and Henry's law, after linearization we obtain: for the gas phase

$$\begin{aligned} \frac{\partial^2 P_1}{\partial x^2} = & -\frac{1}{c_1^2} \frac{\partial^2 P_1}{\partial t^2} + \frac{D}{c_1^2} \frac{\partial^3 P_1}{\partial t \partial x^2} + \frac{T_0 \rho_{at}}{P_{at} T_{10}} \times \\ & \times \left[P_{10} + \frac{32}{d^2} \cdot \left(\vartheta_1 x + \frac{\vartheta_2 - \vartheta_1}{T_{10} - T_{20}} (T_0 - T_c) \alpha \frac{x^2}{2} \right) \cdot Q_0 \cdot \left(1 - \left(\frac{T_c}{T_{10}} - 1 \right) \alpha x \right) \right] \times \\ & \times k \frac{\partial^2 P_1}{\partial t^2} + \frac{T_0 \rho_{at}}{P_{at} T_{10}} \cdot \left[1 - \left(\frac{T_c}{T_{10}} - 1 \right) \alpha x \right] \cdot \frac{\partial P_1}{\partial t} \cdot k \cdot \frac{\partial P_1}{\partial t} + \frac{32}{d^2} \times \\ & \times \left[\vartheta_1 + \frac{\vartheta_2 - \vartheta_1}{T_{10} - T_{20}} (T_0 - T_c) \alpha x \right] \cdot \left\{ -\frac{1}{c_1^2} \frac{\partial P_1}{\partial t} + \frac{D}{c_1^2} \frac{\partial^2 P_1}{\partial x^2} + \frac{T_0 \rho_{at}}{P_{at} T_{10}} \times \right. \\ & \left. \left[P_{10} + \frac{32}{d^2} \left(\vartheta_1 x + \frac{\vartheta_2 - \vartheta_1}{T_{10} - T_{20}} (T_{10} - T_c) \alpha \frac{x^2}{2} \right) Q_0 \left(1 - \left(\frac{T_c}{T_{10}} - 1 \right) \alpha x \right) \right] k \frac{\partial P_1}{\partial t} \right\} + \end{aligned}$$

$$\begin{aligned}
& +Q_{10}\frac{32}{d^2}\left[\frac{\vartheta_2-\vartheta_1}{T_{10}-T_{20}}(T_{10}-T_c)\alpha\right]+V_{10}\left\{-\frac{1}{c_1^2}\frac{\partial^2 P_1}{\partial x\partial t}+\frac{D}{c_1^2}\frac{\partial^3 P_1}{\partial x^3}+\frac{T_0\rho_{at}}{P_{at}T_1}\times\right. \\
& \times\left[1-\left(\frac{T_c}{T_{10}}-1\right)\alpha x\right]\cdot\frac{32}{d^2}\cdot\left[\vartheta_1x+\frac{\vartheta_2-\vartheta_1}{T_{10}-T_{20}}(T_{10}-T_c)\alpha x\right]Q_0k\frac{\partial P_1}{\partial t}+\frac{T_0\rho_{at}}{P_{at}T_{10}}\times \\
& \times\left[-\left(\frac{T_c}{T_{10}}-1\right)a\right]\cdot\left[P_{10}+\frac{32}{d^2}\cdot\left(\vartheta_1x+\frac{\vartheta_2-\vartheta_1}{T_{10}-T_{20}}(T_{10}-T_c)\cdot\alpha\cdot\frac{x^2}{2}\right)\cdot Q_0\right]\times \\
& \times k\cdot\frac{\partial P_1}{\partial t}+\frac{T_0\rho_{at}}{P_{at}T_{10}}\left[1-\left(\frac{T_c}{T_{10}}-1\right)\alpha x\right]\left[P_{10}+\frac{32}{d^2}\cdot\left(\vartheta_1x+\frac{\vartheta_2-\vartheta_1}{T_{10}-T_{20}}\times\right.\right. \\
& \times(T_{10}-T_c)\cdot\alpha\cdot\frac{x^2}{2}\Big)\cdot Q_0\Big]k\cdot\frac{\partial^2 P_1}{\partial x\partial t}+\frac{T_0\rho_{at}}{P_{at}T_{10}}\left[-\left(\frac{T_c}{T_{10}}-1\right)\alpha\right]\left[P_{10}+\frac{32}{d^2}\times\right. \\
& \times\left(\vartheta_1x+\frac{\vartheta_2-\vartheta_1}{T_1-T_2}(T_1-T_c)\cdot\alpha\cdot\frac{x^2}{2}\right)\cdot Q_0\Big]k\frac{\partial P_1}{\partial t}++\frac{T_0\rho_{at}}{P_{at}T_{10}}\left[1-\left(\frac{T_c}{T_{10}}-1\right)\alpha x\right] \\
& \left.\left[P_{10}+\frac{32}{d^2}\cdot\left(\vartheta_1x+\frac{\vartheta_2-\vartheta_1}{T_1-T_2}(T_1-T_c)\cdot\alpha\cdot\frac{x^2}{2}\right)\cdot Q_0\right]k\frac{\partial^2 P_1}{\partial x\partial t}\right\} \quad (3)
\end{aligned}$$

For liquid

$$\begin{aligned}
& \frac{\partial^2 P_2}{\partial x^2}=-\frac{1}{c_2^2}\frac{\partial^2 P_2}{\partial t^2}+\frac{D}{c_1^2}\frac{\partial^3 P_2}{\partial t\partial x^2}+\vartheta_mk\frac{\partial^2 P_2}{\partial x\partial t}+\frac{32}{d^2}\times \\
& \times\left[\vartheta_{1m}+\frac{\vartheta_{2m}-\vartheta_{1m}}{T_{10}-T_{20}}(T_{10}-T_c)\cdot\alpha\cdot x\right]\left[-\frac{1}{c_2^2}\frac{\partial\rho_2}{\partial t}+\frac{D}{c_2^2}\frac{\partial^2 P_2}{\partial x^2}+\rho_{2m}k\frac{\partial P_2}{\partial t}\right]+ \\
& +\frac{32}{d^2}\frac{\vartheta_{2m}-\vartheta_{1m}}{T_{10}-T_{20}}\cdot Q_0+V_{20}\left[-\frac{1}{c_1^2}\frac{\partial^2 P_2}{\partial t\partial x}-\frac{D}{c_2^2}+\rho_mk\frac{\partial^2 P_2}{\partial x\partial t}\right] \quad (4)
\end{aligned}$$

From the obtained equations, taking into account the initial and boundary conditions, it is possible to determine P_1 and P_2 and, on this basis, Q_1 and Q_2 .

References

- [1] Shoham O. Mechanistic Modeling of Gas-Liquid Two-Phase Flow in Pipes. Richardson: Society of Petroleum Engineers (SPE), 2006.
- [2] Ishii M., Hibiki T. Thermo-Fluid Dynamics of Two-Phase Flow. 2nd ed. New York: Springer, 2011.

Modeling the Dynamics of a Gas-Liquid Mixture Flow in a Pipe, Taking Into Account Phase Transformations and Pipe Wall Deformation

Elkhan Abbasov, Serkar Imamaliyev

Institute of Mathematics, Baku, Azerbaijan
(e-mail: aelhan@mail.ru, serkar.imamaliyev@mail.ru)

When transporting a gas-liquid mixture under high pressure through a pipeline, the pipe wall deforms, which affects the dynamics of the mixture. The hydrodynamics of gas-liquid flows with deformable walls involves complex fluid-structure interactions, where fluid forces (such as pressure and turbulence) cause structural deformations, which in turn alter the flow, affecting bubble dynamics, shock waves, and energy dissipation. This is crucial in systems such as pipelines, turbomachines, and underwater acoustics. Failure to consider this factor can sometimes lead to incorrect conclusions. Therefore, the development of a simplified model of the dynamics of gas-liquid mixture flow in a pipe with a deformable wall suitable for practical application is of both scientific and practical interest.

Let us consider the movement of a gas-liquid mixture in a pipe with a deformable wall.

The equation of continuity of the flow of a gas-liquid mixture in a simplified model taking into account phase transformations has the form

$$-\frac{1}{c_1^2} \frac{\partial p}{\partial t} = \frac{\partial Q_1}{\partial x} - \frac{D_1}{c_1^2} \frac{\partial^2 p}{\partial x^2} - \frac{p}{p_{at}} \rho_{at} k \frac{\partial p}{\partial t} + \frac{p}{p_{at} f_0} \rho_{at} [x_{i0} - k(p - p_0)] 2\pi R \frac{\partial w}{\partial t} \quad (1)$$

As a first approximation, neglecting the friction force between the phases for the equation of motion of the gas phase, we will have

$$-\frac{\partial p}{\partial x} = \frac{\partial Q_1}{\partial t} + Q_1 \frac{2\pi R}{f_0} \frac{\partial w}{\partial t} + 2aQ_1 + p \frac{2\pi R}{f_0} \frac{\partial w}{\partial x} + v_{10} \frac{\partial Q_1}{\partial x} \quad (2)$$

Differentiating equation (1) with respect to t , and equation (2) with respect to the x -coordinate, and subtracting one equation from the other, taking into account Henry's law, we obtain

$$-\frac{1}{c_1^2} \frac{\partial p}{\partial t^2} + \frac{\partial^2 p}{\partial x^2} = -\rho_0 k \frac{\partial^2 p}{\partial t^2} + 2\pi R \frac{\partial^2 w}{\partial t^2} \frac{\rho_0}{f_0} [x_{1,0} - k(p - p_0)] - 2\pi R \frac{\partial w}{\partial t} \frac{\rho_0}{f_0} k \frac{\partial p}{\partial t} -$$

$$\begin{aligned}
& -\frac{D}{c_1^2} \frac{\partial^3 p}{\partial x^2 \partial t} - Q_1 \frac{2\pi R}{f_0} \frac{\partial^2 w}{\partial t \partial x} - 2a \left(-\frac{1}{c_1^2} \frac{\partial p}{\partial t} + \rho_0 k \frac{\partial p}{\partial t} - \right. \\
& -2\pi R \frac{\partial w}{\partial t} \frac{\rho_0}{f_0} [x_{1,0} - k(p - p_0)] + \left. \frac{D}{c_1^2} \frac{\partial^2 p}{\partial x^2} \right) - p \frac{2\pi R}{f_0} \frac{\partial^2 w}{\partial x^2} - \\
& -v_{1,0} \left(-\frac{1}{c_1^2} \frac{\partial^2 p}{\partial t \partial x} + \rho_0 k \frac{\partial^2 p}{\partial t \partial x} - 2\pi R \frac{\partial^2 w}{\partial t \partial x} \frac{\rho_0}{f_0} [x_{1,0} - k(p - p_0)] + \right. \\
& \left. + 2pR \frac{\partial w}{\partial t} \frac{\rho_0}{f_0} k \frac{\partial p}{\partial x} + \frac{D}{c_1^2} \frac{\partial^3 p}{\partial x^3} \right) \quad (3)
\end{aligned}$$

Initial and boundary conditions:

$$p|_{t=0} = p_1(0) - \frac{p_1(0) - p_2(0)}{l} \cdot x, \quad 0 < x \leq l \quad (4)$$

$$\frac{\partial p}{\partial t} \Big|_{t=0} = 0, \quad 0 < x = l; \quad (5)$$

$$p|_{x=0} = p_1(t), \quad t > 0; \quad (6)$$

$$p|_{x=l} = p_2(t), \quad t > 0; \quad (7)$$

$$\frac{\partial^2 p}{\partial x^2} \Big|_{x=0} = 0, \quad t > 0 \quad (8)$$

The deformation w of the pipe wall is determined from the equation

$$\frac{\partial^2 w(x, t)}{\partial t^2} + \frac{D}{\rho \delta} \frac{\partial^4 w(x, t)}{\partial x^4} + \frac{2E}{\rho \pi R^2} w(x, t) = \frac{2}{\rho \pi \delta} p(x, t) \quad (9)$$

We will seek a solution to equation (3) satisfying the boundary conditions (4)-(8) in the form:

$$p(x, t) = p_1(t) - \frac{p_1(t) - p_2(t)}{l} x + \sum_{i=1}^{\infty} \phi_i(t) \sin \frac{i\pi x}{l} \quad (10)$$

where $\phi_i(t)$ is an unknown function dependent on time t .

By determining $w(x, t)$ from (9) and substituting it into equation (3), we obtain a differential equation for $p(x, t)$. Next, by substituting expression (10) into the resulting equation, we can determine $\phi_i(t)$. Substituting $\phi_i(t)$ into expression (10), we obtain the pressure field in the pipe, taking into account phase transitions and pipe wall deformation as the gas-liquid mixture moves into it.

References

- [1] Lykov AV, Mikhailov Yu. A., Theory of Heat and Mass Transfer, Moscow, 1963
- [2] Collection of articles edited by S. S. Kutateladze. Heat Transfer and Hydraulics of Two-Phase Media. Moscow, 1961, 392 p.
- [3] Wallis. One-Dimensional Two-Phase Flow. Moscow, Mir, 1972
- [4] Levich V. G. Physicochemical Hydrodynamics. USSR Academy of Sciences Press, 1972
- [5] Landau L. D., Lifshchits E. M. Hydrodynamics. Moscow, Nauka, 1986
- [6] Charny I. A. Unsteady Motion of a Real Fluid in Pipes. Moscow, Nedra, 1975, p. 224
- [7] Leibenzon L. S. Collected Works. Volume II. Underground Hydrogas Dynamics. Moscow, 1953, 544 p.
- [8] Nigmatulin R.I. Dynamics of Multiphase Media. Moscow: Nauka, 1987, 345 p.
- [9] Akulichev V.A. Cavitation in Cryogenic and Boiling Liquids. Moscow: Nauka, 1978
- [10] Kutataladze S.S. Heat Transfer during Condensation and Boiling. Moscow: Matgiz, 1952
- [11] Fayzullaev D.F., Umarov A.I., Shakirov A.A. Hydrodynamics of Single- and Two-Phase Media and Its Practical Applications. Tashkent: FAN Publishing House of the Uzbek SSR, 1980, 163 p.

On the Wave Propagation in the Eccentric Hollow Cylinder Containing an Inviscid Compressible Fluid

Surxay Akbarov^a, Emin Bagirov^b, Fuad Huseynov^b,
Arzu Abbasova^b

^a*Yildiz Technical University, Istanbul, Türkiye*

^b*Institute of Mathematics, Baku, Azerbaijan*

(e-mail: akbarov@yildiz.edu.tr, bagirliemin1@gmail.com)

Introduction. Dynamic problems related to cylinders with eccentric cross-section are relatively difficult to solve, and the results of studies of these problems have great significance for various branches of modern industry, such as shipping, the nuclear industry, aerospace, civil and military engineering, and many others. Nevertheless, so far, however, there have only been a few studies on the dynamics of the eccentric hollow cylinder. The attempts in this field were made in the paper Shah (1973), Hasheminejad and Mirzaei (2009), Hasheminejad and Mousavi-Akbarzadeh (2012). In the paper Akbarov and Farajov (2026) the method for solving dynamic problems for the eccentric hollow cylinder, according to which, differ from the methods employed in the framework of the above works.

However, in the paper Akbarov and Farajova (2025) relate to the empty hollow cylinder. In the present paper we attempt to develop the method presented in the paper Akbarov and Farajova (2025) to the case where the eccentric hollow cylinder containing the compressible inviscid fluid.

Formulation of the problem and solution method. Consider a circular cylinder of infinite length and assume that this cylinder contains an eccentric cylindrical hole of also infinite length and this hole is filled with the compressible inviscid fluid (Fig. 1a). Assume that the central axes of the cylinder and the cylindrical hole are parallel to each other and that the distance between these axes is R_{cb} . We connect the coordinates of the cylindrical $O r \theta z$ and the Cartesian $O x_1 x_2 x_3$ system with the central axis of the cylinder. Moreover, assume that $z = x_3$ and the center of the cylindrical hole is on the coordinate axis $O x_1$ and the distance between this center and the chosen origin of the coordinate system is R_{cb} .

We also assume that the cross-section of this hole is a circle and the radius of this circle is R_b . At the same time, we suppose that the cross-section

of the cylinder is a double-connected eccentric circle which occupies the area entering the circle of radius R_c ($R_c > R_b$), from which a circle of radius R_b is subtracted (Fig. 1b). We also associate the Cartesian $O_0x_{10}x_{20}x_{30}$ and cylindrical $O_0r_0\theta_0z_0$ coordinate systems with the central axis of the cylindrical hole. The connection between the selected coordinate systems are the following relations.

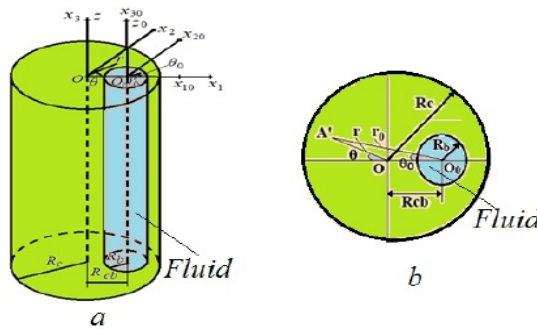


Fig. 1. The sketch of the eccentric hollow cylinder contained a fluid and selected coordinate systems (a) and the geometry of the cross section of this cylinder (b)

$$x_1 = R_{cb} + x_{10}; x_2 = x_{20}; x_3 = x_{30}; r \exp(i\theta) = R_{cb} + r_0 \exp(i\theta_0), z = z_0. \quad (1)$$

Following the above preparatory procedure, we write the field equations of the three-dimensional linear theory of elastic waves in the cylindrical coordinate system in the case where the material of the cylinder is isotropic and homogeneous. These equations can be presented formally as equations of motion $\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2}, \dots$, elasticity relations: $\dots$, Strain-displacement relations: $\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \dots$.

The flow of the fluid filled the hole is described by the following linearized Euler equations.

Continuity equation, the rheological relations and state equation. We add to these equations the corresponding boundary and compatibility conditions

This completes the problem formulation. Under solution to the formulated problem the method proposed in the work Akbarov and Farajova (2025) is developed and employed.

Thus, after satisfying the conditions listed above we obtain the infinite system linear algebraic equations, according to which, we determine the dispersion equation which is solved numerically.

In the work it has been also obtained concrete numerical results illustrated the influence of the eccentricity of the cylinder on the dispersion curves.

References

- [1] S.D. Akbarov, S. S. Farajova, The method for solving 3D problems of the eccentric hollow cylinder and its application to the study of the wave dispersion propagating in this cylinder. *Structural Engineering and Mechanics*, Vol. 96, No. 4 (2025) 000-000
- [2] Hasheminejad, S.M. and Mirzaei, Y. (2009), "Free vibration analysis of an eccentric hollow cylinder using exact 3D elasticity theory", *J. Sound Vib.*, 326, 687-702.
- [3] Hasheminejad, S.M. and Mousavi-Akbarzadeh, H. "Vibroacoustic response of an eccentric hollow cylinder", *J. Sound Vib.*, 331, (2012), 3791-3808.
- [4] Shah, A.H. , "Elastic waves in a circular cylinder with an eccentric bore", *Acustica*, **28**(3), (1973) 251-258.

On Some a Spects of Mathematical Modeling of Heat Conduction in Nonequilibrium Multicomponent Media

Ramil Bakhtizin^a, Geylani Panahov^b, Fuad Khudaverdiyev^b

^a*Ufa State Oil Technical University, Kosmonavtov St. 1, Ufa 450064, Russia*

^b*Institute of Mathematics, Baku, Azerbaijan*
(e-mail: gpanahov@gmail.com)

Development of new technologies for oil production and refining, for creation of new composite materials, heat exchange equipment and energy systems, necessitates the description of heat conduction processes occurring in media with complicated internal structure [1, 2]. Such media include high

viscosity oils, polymer solutions, disperse systems, nanofluids and multicomponent composite materials, where dissolution processes, structural transformations, phase transitions and chemical reactions occur simultaneously. In such circumstances, the classical theory of heat conduction based on Fourier's law does not always provide an adequate description of the process, because this theory suggests the instant propagation of heat perturbation and the validity of the local thermodynamic equilibrium hypothesis [4, 6, 7].

In this work, we offer a new approach to constructing mathematical model of heat conduction in nonequilibrium multicomponent media based on Gyarmati's variational principle (the principle of least dissipation of energy) [3, 5]. This approach allows to consider relaxation character of internal physical and chemical processes and to obtain a closed system of evolution equations describing simultaneously both heat conduction and structural transformations inside the medium. As an initial relation, we consider the following kinetic relaxation equation of variations of internal structural parameter:

$$\frac{d\xi}{dt} = f(\xi, F),$$

where ξ characterizes the generalized state of internal structure of the medium (degree of dissolution, destruction, or phase transition), and F is a totality of all external influences. Such an approach makes it possible to describe a wide class of processes regardless of the physical nature of internal transformations.

Macroscopic description of the system is based on the Gibbs fundamental equation in entropic form and Gyarmati's variational principle [3] expressed by the formula $\delta \int_V (\sigma - \Psi) dV = 0$. Applying variational approach and energy conservation law, we obtain a system of relaxation equations of heat conduction, which establishes relationship between heat flow and internal structural transformations [5, 8, 9]. By Gauss-Ostrogradsky theorem and the linear Onsager relations, this system becomes

$$\begin{aligned} q &= -\lambda \nabla T \\ \tau_\xi \frac{dA}{dt} &= -A + \chi T \end{aligned}$$

where λ is a heat transfer coefficient, τ_ξ is a characteristic relaxation time of the structure, and χ is a thermodynamic coefficient of connectivity.

As the internal structural variable is a state function $\xi = \xi(A, T)$, the total differential $\frac{d\xi}{dt}$ is the basis for the phenomenological theory of heat and mass transfer:

$$\rho c_v \frac{\partial T}{\partial t} + \nabla \cdot q = Q_v \frac{\partial \xi}{\partial t},$$

$$\tau_\xi \frac{\partial \xi}{\partial t} + \xi = \xi_0(T),$$

where c_v is a specific heat capacity, Q_v is an effect of the thermal impact of an internal phenomenon, and τ is a macroscopic relaxation time.

Based on the relaxation formalism of the thermodynamics of irreversible processes and the Gyarmati's variational principle, the suggested method allows to uniformly describe heat conduction processes in the media with internal structure. The general heat conduction equation of hyperbolic type successfully solves the paradox of heat propagation at an infinite speed, characteristic for the classical parabolic equation.

Stable solution methods have been well developed for the inverse heat conduction problems regarding this class of differential equations. These are of practical interest in design and production of heat exchangers, oil trunk pipelines, dispersed materials processing systems, other complex engineering problems of materials science and heat engineering.

References

- [1] G. Capriz, K. Wilmanski, and P. M. Mariano, *Exact and approximate Maxwell–Cattaneo-type descriptions of heat conduction: A comparative analysis*, International Journal of Heat and Mass Transfer, vol. 175, Art. no. 121362, 2021.
- [2] S. R. de Groot and P. Mazur, *Non-Equilibrium Thermodynamics*, North-Holland Publishing Company, 1962.
- [3] I. Gyarmati, *Non-Equilibrium Thermodynamics: Field Theory and Variational Principles*, Springer, 1974.
- [4] D. Jou, J. Casas-Vázquez, and G. Lebon, *Extended Irreversible Thermodynamics*, 4th ed. Springer, 2010.
- [5] Jou and L. Restuccia, *Non-equilibrium thermodynamics of heat transport in superlattices, graded systems, and thermal metamaterials with defects*, Entropy, vol. 25, no. 7, Art. no. 1091, 2023.

- [6] R. Kovács, *Heat equations beyond Fourier: From heat waves to thermal metamaterials*, NoPhysics Reports, vol. 1048, pp. 1–75, 2024.
- [7] C. F. Munafò, P. Rogolino, and R. Kovács, *Nonlinear thermal analysis of two-dimensional materials with memory*, International Journal of Heat and Mass Transfer, vol. 219, Art. no. 124847, 2024.
- [8] B. Straughan, V. Tibullo, and F. Passarella, *NoBuoyancy-driven convection with a Cattaneo flux model*, NoJournal of Non-Equilibrium Thermodynamics, vol. 49, no. 2, pp. 215–220, 2024.
- [9] S. Varshegaa, E. Ragulkumar, F. Peter, K. Loganathan, and P. Sambath, *NThermal and mass transfer performance of non-Newtonian fluids under MHD and chemical reaction effects with thermophoresis and Brownian motion*, NoDiscover Applied Sciences, vol. 7, Art. no. 748, 2025.

Dynamics of an Initially Stressed Eccentric Hollow Cylinder

Sona Farajova, Yusif Sevdimaliyev

Baku State University, Baku, Azerbaijan

(e-mail: ferecovasona97@gmail.com, yusifsev@gmail.com)

Introduction. Solving dynamic problems for eccentric hollow cylinders is challenging, yet critically important for shipbuilding, nuclear energy, aerospace, and engineering applications, despite limited existing research [1-4]. The dispersion characteristics of waves propagating along the Oz-axis in an eccentric hollow cylinder are investigated within the following mathematical model: the governing equations of three-dimensional linear theory in cylindrical coordinates, a generalized Hooke's law expressing the relationship between the components of the stress and strain tensors for an isotropic material, the Cauchy relations describing the displacement-strain dependence within the framework of small deformations, as well as the corresponding boundary conditions.

Formulation of the problem. If we relate the Cartesian $O_0x_{10}x_{20}x_{30}$ and cylindrical $O_0r_0\theta_0z_0$ coordinate systems to the center axis of the cylindrical hole, the relationship between the chosen coordinate systems would be as

follows:

$$x_1 = R_{cb} + x_{10}; x_2 = x_{20}; x_3 = x_{30}; r \exp(i\theta) = R_{cb} + r_0 \exp(i\theta_0), z = z_0. \quad (1)$$

Boundary conditions can be expressed in the following form for the cases $r = R_c$ and $r_0 = R_b$, respectively in the cylindrical coordinate systems $O r \theta z$ and $O_0 r_0 \theta_0 z_0$:

$$\begin{aligned} \sigma_{rr}|_{r=R_c} &= 0, \sigma_{r\theta}|_{r=R_c} = 0, \sigma_{rz}|_{r=R_c} = 0, \\ \sigma_{rr}|_{r_0=R_b} &= 0, \sigma_{r\theta}|_{r_0=R_b} = 0, \sigma_{rz}|_{r_0=R_b} = 0. \end{aligned} \quad (2)$$

Method of solution. According to [5], potentials are given, through which the displacements are presented as follows:

$$\begin{aligned} u_r &= \frac{\partial}{r \partial \theta} \Psi - \frac{\partial^2}{\partial r \partial z} X, u_\theta = -\frac{\partial}{\partial r} \Psi - \frac{\partial^2}{r \partial \theta \partial z} X, \\ u_z &= (A_{13} + A_{44})^{-1} (A_{11} \Delta_1 + A_{44} \frac{\partial^2}{\partial z^2} - \rho \frac{\partial^2}{\partial t^2}) X, \\ \Delta_1 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \end{aligned} \quad (3)$$

By substituting the solution form corresponding to wave propagation along the Oz axis into the governing equations for these potentials, the following ordinary differential equations are obtained for the functions $\Psi_n^0(r_0)$ and $X_n^0(r_0)$.

$$\begin{aligned} (\Delta_{1n}^0 + \zeta_1^2) \Psi_n^0 &= 0, (\Delta_{1n}^0 + \zeta_2^2) (\Delta_{1n}^0 + \zeta_3^2) X_n^0 = 0, \\ \Delta_{1n}^0 &= \frac{\partial^2}{\partial r_0^2} + \frac{1}{r_0} \frac{\partial}{\partial r_0} - \frac{n^2}{r_0^2}. \end{aligned} \quad (4)$$

Numerical results for the dispersion curves of waves propagating along the axis in an eccentric cylinder are considered. The material is assumed to be isotropic, and the initial stresses are zero. The eccentricity is characterized by the ratio R_{cb}/R_c .

These results show that at relatively small values of the ratio R_{cb}/R_c (e.g., 0.15), the eccentricity of the hollow cylinder leads to the formation of new dispersion curves, as depicted in Figure 1(a). A similar effect is observed for all values of $R_{cb}/R_c < 0.17$. However, when $R_{cb}/R_c \geq 0.18$, the situation changes: in this case, the eccentricity leads to the formation of new types of dispersion

curves, labeled Mode $(1+2)_1$ and Mode $(1+2)_2$ in Figure 1(b). A comparison between Figure 1(a) and Figure 1(b) shows that as R_{cb}/R_c increases, Mode $(1+2)_1$ and Mode $(1+2)_2$ in Figure 1(a) touch each other, resulting in new types of dispersion curves, labeled Mode $(1+2)_1$ and Mode $(1+2)_2$.

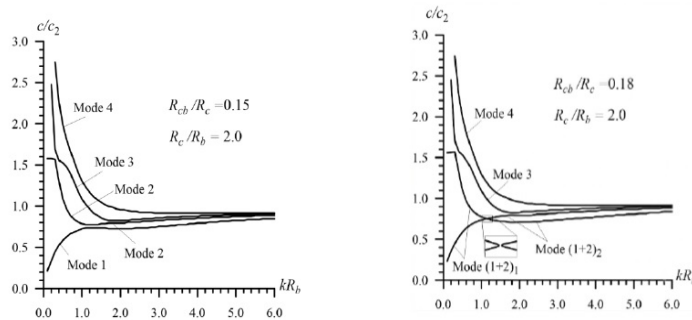


Fig. 1. Dispersion curves obtained in the case where (a) $R_{cb}/R_c = 0.15$ and (b) $R_{cb}/R_c = 0.18$

As a result, we can note that, Thus, the conducted numerical studies show that the eccentricity of the hollow cylinder has a significant effect on the qualitative and quantitative characteristics of the dispersion curves and leads to the formation of new types of dispersion regimes depending on the value of the parameter R_{cb}/R_c .

References

- [1] S.D. Akbarov, Dynamics of Pre-strained Bi-material Elastic Systems: Linearized Three-Dimensional Approach, *Springer, Heidelberg, New York*, 2015.
- [2] S.D. Akbarov, S.S. Farajova, The method for solving 3D problems of the eccentric hollow cylinder and its application to the study of the wave dispersion propagating in this cylinder. *Structural Engineering and Mechanics*, 96(4), 2025, 257-270.
- [3] S.D. Akbarov, S.S. Farajova, Y.M. Sevdimaliyev, On the solution method of the dynamics of the eccentric hollow cylinder made of transversally isotropic

material with homogeneous initial stresses. *In Proceedings of the 9th International conference on Control and Optimization with industrial Applications*. 27 - 29, August, 2024, Istanbul, Türkiye, pp. 629- 633.

- [4] S.D. Akbarov, S.S. Farajova, Y.M. Sevdimaliyev, On the dispersion of longitudinal waves propagating in the eccentric hollow cylinder. International Congress on new trends in mechanics Azerbaijan State Pedagogical University, 2-5 September 2025, Baku.
- [5] A.N. Guz, Elastic Waves in Bodies with Initial Stresses, A.C.K. Kyiv, 2004

Assessment of the Technical Condition of Mechanical Systems Using Digital Twin and Artificial Intelligence Technologies

**Elshan Hashimov^{a,b}, Bayram Ibrahimov^{a,b}, Marko Radovanovic^c,
Aziz Talibov^a, Valeh Bakhshali^a, Ismayil Ismayil^a**

^a*Azerbaijan Technical University, Baku, Azerbaijan*

^b*National Defense University, Baku, Azerbaijan*

^c*University of Defence, Belgrade, Serbia*

(e-mail: hasimovel@gmail.com, bayram.ibrahimov@aztu.edu.az,
markoradovanovicgdb@yahoo.com, talibovaziz65@gmail.com,
valeh.bakhshali@aztu.edu.az, ismayil.ismayil@aztu.edu.az)

Mechanical equipment in industry, transport, energy, and automated production operates under changing loads and is continuously affected by vibration, friction, heating, and material ageing. Most faults do not occur without warning. Corrective maintenance responds only after a failure, whereas maintenance at fixed intervals may replace serviceable components too early or overlook rapidly developing damage. Digital twins combined with artificial intelligence offer a more responsive alternative.

This study develops a compact assessment framework that combines a physics-based digital twin, multisensor monitoring, and machine-learning methods. A representative rotating mechanical system consisting of an electric motor, coupling, shaft, bearing assemblies, gearbox, and actuating mechanism is considered. Its technical condition is described through six groups of

data: vibration, temperature, deformation and stress, rotational and loading parameters, acoustic signals, and lubricant condition. Vibration signals are processed in the time, frequency, and time-frequency domains. Temperature, strain, acoustic-emission, and oil-condition measurements provide additional evidence of increasing friction, structural weakening, impact events, and surface wear.

The digital twin is constructed from the geometric, inertial, stiffness, damping, friction, transmission, clearance, and loading parameters of the physical system. Its general dynamic behaviour can be represented as

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F(t) \quad (1)$$

During operation, the model parameters are periodically recalibrated using sensor measurements. The difference between the measured response and the response calculated by the digital twin is defined as a residual signal:

$$r_i(t) = x_i^f(t) - x_i^m(t) \quad (2)$$

where $x_i^f(t)$ and $x_i^m(t)$ denote the measured and modelled responses, respectively.

For a healthy system, the residual remains within limits determined by measurement uncertainty and modelling error. A sustained increase in residual magnitude, or a change in its statistical characteristics, may indicate wear, stiffness loss, excessive backlash, imbalance, or a sensor malfunction. Rather than relying on a single signal, the assessment is organised as a closed diagnostic loop. The measurements are filtered and normalised, the digital model is updated, simulated and measured responses are compared, and the artificial-intelligence module identifies anomalies, fault classes, and degradation trends. Supervised algorithms are suitable when labelled fault examples are available. When such data are limited, clustering, autoencoders, and one-class methods provide more practical alternatives [2].

Because the diagnostic variables have different units and ranges, they are normalised to the interval from zero to one and combined into an integral technical-condition index:

$$K_{\text{tech}} = \sum_{i=1}^6 w_i K_i \quad (3)$$

In this expression, K_i is a normalised diagnostic indicator and w_i is its relative weight. The weights are non-negative and their sum is equal to one. Values of the integral index close to one indicate normal operation, while a gradual decrease reflects progressive deterioration. The weights can be assigned through expert judgement, the analytic hierarchy process, entropy-based methods, or statistical identification. The index should be interpreted together with the residual signals rather than used as an isolated scalar measure.

The condition index changes with operating time, loading regime, environmental influence, and maintenance history. If degradation is locally close to linear, the remaining useful life may be estimated as

$$T_{\text{RUL}} \approx \frac{K_{\text{tech}}(t) - K_{\text{cr}}}{v_d(t)} \quad (4)$$

where K_{cr} is the critical condition threshold and v_d is the current degradation rate. When deterioration is nonlinear, forecasting should rely on time-series methods, recurrent neural networks, long short-term memory networks, or probabilistic models. This structure supports a gradual shift from reactive maintenance to condition-based and predictive maintenance, with intervention scheduled according to the actual equipment state and its expected degradation trajectory [3].

For illustration, five supervised classifiers were compared at the diagnostic stage. The linear support vector machine produced the highest overall accuracy at 96.37%, followed by the decision tree at 95.76% and the random forest at 94.92%. The radial-basis support vector machine and k-nearest-neighbour models achieved 70.40% and 71.47%, respectively. Three condition clusters were identified with 100% accuracy, while the most difficult cluster was classified with 96.50% accuracy. The comparison shows that diagnostic performance is determined not only by the selected algorithm, but also by the physical relevance of the input features and the quality of the data supplied by the digital twin.

Taken together, the digital-twin residuals and multisensor indicators make it possible to detect weak fault signatures at an early stage, combine heterogeneous diagnostic evidence, and estimate both degradation rate and remaining useful life. Reliability in practical use will depend on sensor placement, sampling frequency, calibration, model fidelity, and the interpretability of artificial-intelligence decisions. The framework can be adapted to gearboxes, bearings, electric drives, pumps, turbines, and other rotating mechanical systems.

References

- [1] F.K. Moghadam, G.F.S. Reboucas, A.R. Nejad, *Digital twin modeling for predictive maintenance of gearboxes in floating offshore wind turbine drive-trains*, Forschung im Ingenieurwesen, 2021, 85, 273-286.
- [2] Y. Xu, Y. Sun, X. Liu, Y. Zheng, *A digital-twin-assisted fault diagnosis using deep transfer learning*, International Journal of Intelligent Systems, 2021, 36(8), 3688-3709.
- [3] D. Zhong, Z. Xia, Y. Zhu, J. Duan, *Overview of predictive maintenance based on digital twin technology*, Heliyon, 2023, 9(4), e14534.

Dynamics of Flexural-Gravity-Capillary Waves in Floating Viscoelastic Plates under Stochastic Loading

Aynura Khudiyeva

Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
(e-mail: aynur-khudiyeva@rambler.ru)

The interaction between floating viscoelastic plates and incompressible fluids is important in marine engineering and offshore structures. Random external loading significantly affects wave propagation, damping, and resonance.

Mathematical Formulation. The mathematical expectation of the stochastic load is defined as:

Consider a viscoelastic plate subjected to a non-stationary random force $P(\xi_1, \xi_2, t)$, where t -denotes time and (ξ_1, ξ_2) -represents a random spatial point. The mathematical expectation (mean value) of a function $P(\xi_1, \xi_2, t)$ of two random variables ξ_1, ξ_2 is defined by the formula [1]

$$MP(\xi_1, \xi_2, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\xi_1, \xi_2) P(\xi_1, \xi_2, t) d\xi_1 d\xi_2 \quad (1)$$

$$(\xi_1, \xi_2) = \left(\frac{x_i - \xi_i}{d} \right) = \frac{1}{\pi d^2} \exp \left(- \sum_{i=1}^2 \frac{(x_i - \xi_i)^2}{d^2} \right) \quad (2)$$

In the case of a potential flow of an incompressible viscous fluid, the continuity equation and the third equation of motion simplify and take the form [2].

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial z^2} = \left(\Delta_- + \frac{\partial^2}{\partial z^2} \right) \varphi \quad (3)$$

$$-P_2 = \rho \frac{\partial \varphi}{\partial t} + \rho_2 g z \quad (4)$$

The dynamic linearized boundary condition $z = 0$ at the undisturbed contact is determined using the well-known Laplace formula: $P_2^0(w) + P_2|_{z=w} - P_{pl} = -\sigma R^{-1}$

$$-P_{pl} = \rho_2 \frac{\partial \varphi}{\partial t} + \rho_2 g w - \sigma \Delta_- w \quad (5)$$

The kinematic condition at the contact is given by: $z = 0$

$$\frac{\partial w}{\partial t} = \frac{\partial \varphi}{\partial z} \quad (6)$$

where φ -is the velocity potential, ρ_2 - is the fluid density, w - is the plate deflection, σ - γ is the surface tension coefficient, P_2 - is the fluid pressure, and P_{pl} - is the pressure on the lower surface of the plate.

Taking into account (1) and (6) in the equation of motion of the viscoelastic plate, we obtain

$$\begin{aligned} & D \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)^2 Lw + \rho_1 h \frac{\partial^2 w}{\partial t^2} + \rho_2 g w - \\ & - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) w + \rho_2 \frac{\partial \varphi}{\partial t} \Big|_{z=0} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \$ \left(\frac{x_i - \xi_i}{d} \right) P(\xi_i, t) d\xi_i \end{aligned} \quad (7)$$

where $D = Eh^3/(1 - \nu^2)$ - is the flexural rigidity, ρ_1 -is the density, and h -is the thickness of the plate.

Analytical Solution Using Fourier Transform. To solve the governing equations, the space-time Fourier transform is applied. After transformation, the system equations and boundary conditions are simplified into algebraic form and solve the problem defined by equations (4), (6), and (7), we employ the space-time Fourier transform, as defined by formula $(\bar{x} = \{x_1, x_2\}, \bar{k} = \{k_1, k_2\})$

$$\bar{g}(\bar{k}, \omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(\bar{x}, t) e^{-i(\bar{k}\bar{x} + \omega t)} d\bar{x} dt$$

$$\begin{aligned}\bar{\varphi} &= 2C_2(\omega) e^{kH} ch[k(z+h)] \\ F(k, \omega) &= Dk^4 + \sigma k^2 + \rho_2 g - \left(\rho_1 h + \frac{\rho_2}{kth(kH)} \right) \omega^2 - Dk^4 \bar{H}(\omega) \\ \bar{H}(\omega) &= \int_0^\infty e^{-i\omega t} H(\tau) d\tau\end{aligned}$$

Taking into account the integral representation of Bessel functions $J_0(kr) = \frac{1}{2\pi} \int_0^{2\pi} e^{ikr \cos \theta} d\theta$, in the axisymmetric case we have

$$w(r, t) = \frac{1}{(2\pi)^2} \int_0^\infty \int_{-\infty}^\infty \frac{\exp(-d^2 k^2/4) \bar{P}(k, \omega)}{F(k, \omega)} k J_0(kr) e^{i\omega t} dk d\omega \quad (8)$$

Applying the convolution theorem we obtain

$$w(r, t) = \int_0^\infty \int_{-\infty}^\infty f(r - \rho, t - \tau) k P(\rho, t) d\rho d\tau \quad (9)$$

The Green's function represents the response of the viscoelastic plate to stochastic excitation. Axisymmetric assumptions simplify the integral representation through Bessel functions.

Moving Load and Resonance For a moving stochastic load with constant velocity v , the governing equations are transformed into moving coordinates. The plate deflection depends strongly on the roots of the characteristic polynomial.

For a uniformly distributed force $q(\eta) = q_0$, by integrating expression (9) with respect to η and taking into account the evenness of the integrand with respect to k , we obtain

$$\begin{aligned}w &= \frac{2q}{\pi} \int_{-\infty}^\infty \frac{\sin \frac{kl}{2} \cos k(x - vt) \exp(-d^2 k^2/4)}{k F_1(k, \omega)} dk \\ \xi^4 - (\beta^2 \zeta^2 - \gamma^2) \xi^2 + 1 - \zeta^2 &= 0\end{aligned} \quad (10)$$

We then determine the roots of this polynomial, which correspond to the characteristic wave numbers or frequencies that govern the bending behavior of the plate and play a crucial role in analyzing resonance and dynamic response under various loading conditions.

$$\xi_i = \pm \sqrt{\frac{\beta^2 \zeta^2 - \gamma^2}{2}} \left(1 \pm \sqrt{1 - \frac{4(1 - \zeta^2)}{\beta^2 \zeta^2 - \gamma^2}} \right)^{1/2} \quad (11)$$

Numerical analysis shows that resonance occurs when the velocity of the moving load approaches the wave propagation speed. Complex roots correspond to damped oscillations, while real roots lead to amplification effects.

References

- [1] Loitsiansky L.G. Mechanics of Fluids and Gas, 2003.
- [2] Dias F., Milewski P. Journal of Fluid Mechanics, 2017.

Inverse Problem For Colloidal Particle Transport In A Two-Zone Porous Medium With Different Properties

Bakhtiyor Khuzhayorov^{a,b}, Tursunpulat Dzhiyanov^a

^a*Samarkand State University named after Sharof Rashidov, 15, University Blvd., Samarkand, Uzbekistan*

^b*V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent*
(e-mail: t.djiyanov@mail.ru)

The transport of colloidal particles in porous media has attracted considerable attention because of its wide range of applications in groundwater remediation, enhanced oil recovery, membrane filtration, geothermal engineering, and environmental protection. Accurate prediction of colloidal transport is essential for understanding particle migration, permeability reduction, and the evolution of hydraulic properties in porous formations [1].

Most existing mathematical models assume irreversible deposition of colloidal particles on the solid matrix. Although this assumption adequately describes the initial stage of filtration, numerous experimental investigations have shown that retained particles may subsequently detach and return to the flowing suspension under changing hydrodynamic or physicochemical conditions. Consequently, reversible deposition–detachment mechanisms should be incorporated into transport models to achieve more realistic predictions of long – term filtration processes [2].

In heterogeneous porous media, particle transport is strongly influenced by spatial variations in pore structure and surface properties. Regions with different filtration characteristics exhibit different particle retention and release intensities, leading to nonuniform concentration distributions and significant changes in breakthrough behavior. Therefore, the use of two – zone models provides a more realistic representation of colloidal transport than homogeneous formulations [3].

In this study, the transport of colloidal particles in a porous medium is investigated by considering reversible particle retention in both zones. Each zone is characterized by different attachment and detachment rates, reflecting the heterogeneity of the porous structure. Unlike irreversible deposition models, this approach provides a more realistic description of colloidal transport, since irreversible retention mainly occurs during the initial stage of filtration. A one-dimensional mathematical model is solved numerically to evaluate the influence of reversible retention on the transport behavior and concentration distribution of colloidal particles.

We use a similar scheme to [4] for a medium with dual porosity (Fig.1). The first zone, indexed 1 in the notation, has high permeability, while the second zone has low permeability. In each zone, colloidal particle deposition is governed by irreversible nonequilibrium kinetics.

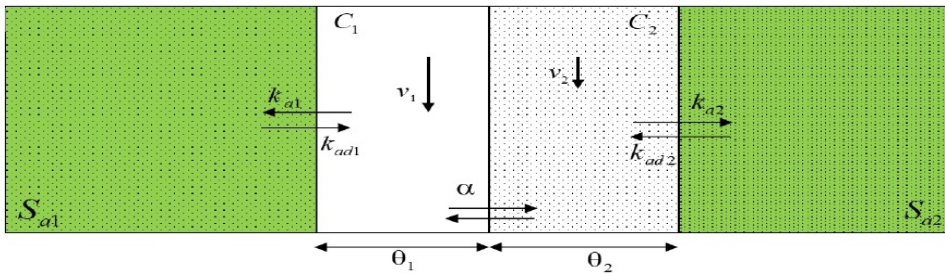


Fig.1. Scheme of solute transport in a two-zone medium

The equations of mass transfer in the one-dimensional case are written in the form [5]

$$\rho \frac{\partial S_{al}}{\partial t} + \theta_l \frac{\partial C_l}{\partial t} = \theta_l D_l \frac{\partial^2 C_l}{\partial x^2} - \theta_l v_l \frac{\partial C_l}{\partial x} + \alpha (C_r - C_l), \quad (1)$$

$$(l = 1; 2; r = 3 - l),$$

where t is time, s , x is distance, m , D_l - coefficient of longitudinal dispersion, m^2/s , v_l - velocity of liquid movement, m/s , $v_1 < v_2$, C_l - volume concentration of the substance in the liquid, S_{al} - concentration of the deposited substance, m^3/kg , θ_l - porosity of zones m^3/m^3 , ρ - density of the medium, kg/m^3 , α - coefficient of mass transfer between zones, s^{-1} .

The deposition of the substance in each zone occurs reversibly in accordance with the kinetic equations

$$\rho \frac{\partial S_{al}}{\partial t} = \theta_l k_{al} C_l - \rho k_{adl} S_{al}, \quad (l = 1, 2), \quad (2)$$

where k_{al} are the coefficients of deposition of a colloidal particles from the liquid phase l onto the solid phase, s^{-1} , k_{adl} are the coefficients of separation of a colloidal particles from the solid phase and transition to liquid, s^{-1} .

Let a fluid with a constant concentration of colloidal particles be pumped into a medium initially saturated with pure liquid from the initial moment of time C_0 . Let us consider periods of time where the concentration field does not reach the right boundary of the medium, $x = \infty$. Under the above assumptions, the initial and boundary conditions for the problem have the form

$$C_l(0, E) = 0, \quad S_{al}(0, x) = 0, \quad (3)$$

$$C_l(t, 0) = C_0, \quad (4)$$

$$\frac{\partial C_l}{\partial x}(t, \infty) = 0, \quad l = 1, 2. \quad (5)$$

Problem (1)-(5) corresponds to the direct statement. Given the coefficients in (1),(2) and (4), the solution is determined $C_l(x, t)$, $S_{al}(x, t)$. When solving the inverse problem, some coefficients in (1),(2) are unknown and must be determined.

Solving the inverse problem requires additional information about the solution to the direct problem. Therefore, we first solve the direct problem with known values of the initial parameters. The additional information here is the concentration values at certain specified points $x_l \in (0, L_1)$ which we will designate as $z_l(t)$.

To investigate the inverse problem of colloidal particles transport, the coefficients k_{a1} and k_{a2} in the system of equations (1), and (2) are determined from additional information obtained from the solution of the corresponding direct problem using the methodology presented in [6–10].

The obtained results demonstrate that incorporating reversible particle retention and zone-dependent kinetic characteristics significantly improves the accuracy of mass transport modeling in two-zone porous media. The proposed numerical approach provides reliable solutions to both the direct and inverse problems, while the accuracy of parameter reconstruction is shown to depend strongly on measurement errors and the choice of initial approximations.

References

- [1] Bear, J. *Dynamics of Fluids in Porous Media*. Dover Publications, New York, 1988.
- [2] Ryan, J. N., & Elimelech, M. "Colloid mobilization and transport in groundwater." *Colloids and Surfaces A: Physicochemical and Engineering Aspects*, 107, 1–56, 1996.
- [3] Berkowitz, B., Cortis, A., Dentz, M., & Scher, H. "Modeling non-Fickian transport in geological formations as a continuous time random walk." *Reviews of Geophysics*, 44(2), RG2003, 2006.
- [4] Leij F.L., Bradford S.A. Colloid transport in dual-permeability media. *Journal of Contaminant Hydrology*. 150. 2013. Pp. 65–76.
- [5] Leij F.J., Bradford S.A. Combined physical and chemical nonequilibrium transport model: analytical solution, moments, and application to colloids. *Journal of Contaminant Hydrology*. 110. 2009. Pp. 87–99.
- [6] Babe G.D., Bondarev E.A., Voevodin A.F., Kanibolotsky M.A. Identification of Hydraulic Models. *Novosibirsk: Nauka, Siberian Branch*, 1980. 161 p.
- [7] Alifanov O. M., Artyukhin E. A., Rumyantsev, S. V. *Extreme Methods for Solving Ill-Posed Problems with Applications to Inverse Heat Transfer Problems*. New York: Begell House Inc., 1995. 288 pp.
- [8] Khuzhayorov B.K., Dzhiyanov T.O., Eshdavlatov Z.Z. Numerical Investigation of Solute Transport in A Non-Homogeneous Porous Medium Using Nonlinear Kinetics. *International Journal of Mechanical Engineering and Robotics Research*, 11(2), 2022, Pp. 79–85.

- [9] Khuzhayorov B.KH, Mustafokulov J.A., Dzhiyanov T.O., Zokirov M.S. Solute Transport with Non-Equilibrium Adsorption in a Non-Homogeneous Porous Medium. *WSEAS Transactions on Fluid Mechanics*, 17, 2022, Pp. 181–188.
- [10] Khuzhayorovich, K.B., Ortikovich, D.T., Bokhodirovich, K.F., Sayfitdinovich, M.S. A Mathematical Modelling of Solid Particles Transport in Two-Zone Porous Media with Non-Linear Kinetics of Deposition. *Mathematical Modelling of Engineering Problems*. Vol. 11. No. 5. May. 2024. Pp. 1293–1300.

Corrosion-Induced Failure of a Circular Concentric Plate Under Internal Pressure

Hijran Mammadova

Institute of Mathematics, Baku, Azerbaijan
(e-mail: hicranmammadova1967@gmail.com)

A circular concentric plate made of an isotropic elastic material is considered. The plate is exposed to an aggressive environment and subjected to internal pressure. The effect of the aggressive environment on the plate is manifested in the form of corrosion uniformly distributed over its entire surface. Such structures are widely used in mechanical engineering, transportation, energy systems, and other branches of industry.

Let the inner radius of the plate be a , the outer radius be b , the internal pressure acting per unit length of the inner circumference be P_a and the plate thickness be $2h$. It is assumed that $h = h(t)$ the plate thickness decreases with time according to the relation where t is time: $h(0) = h_0$, $h(t_*) = 0$, and t_* denotes the time corresponding to complete corrosion wear of the plate. Furthermore, let C be the function characterizing the concentration of the aggressive medium. A cylindrical coordinate system is used: (r, φ, z) . The plane $z = 0$ is taken as the middle surface of the plate: $-h \leq z \leq h$. In this case, the concentration function can be assumed in the form $c(z) = z/h$ for $0 \leq z \leq h$; $c(z) = -z/h$ for $-h \leq z \leq 0$. The above function satisfies the diffusion equation in its stationary form, as well as the corresponding boundary conditions: $c(0) = 0$, $c(\pm h) = 1$.

A plane stress state is assumed in the plate under consideration. Among the components of the stress tensor σ_{ij} ($i, j = r, \varphi, z$), only the radial and circumferential stresses, σ_{rr} , $\sigma_{\varphi\varphi}$, are nonzero. The boundary conditions are given by $\sigma_{rr}(a) = -P_a/h(t)$, $\sigma_{rr}(b) = 0$. It is assumed that the plate boundaries corresponding to the inner radius $r = a$ and the outer radius $r = b$ are protected from corrosion. As a result, corrosion uniformly reduces only the plate thickness, without affecting its inner and outer radii.

The stress components σ_{rr} and $\sigma_{\varphi\varphi}$ are determined by the following expressions:

$$\sigma_{rr} = \frac{a^2 P_a}{(b^2 - a^2) h(t)} \left(1 - \frac{b^2}{r^2} \right) \quad (1)$$

$$\sigma_{\varphi\varphi} = \frac{a^2 P_a}{(b^2 - a^2) h(t)} \left(1 + \frac{b^2}{r^2} \right) \quad (2)$$

The presence of an aggressive environment reduces the strength of the material. The ultimate strength σ_P in the presence of an aggressive medium is determined by the expression: $\sigma_P = \sigma_0 (1 - \gamma c)$, where σ_0 denotes the overall ultimate strength of the material of the plate, and γ — is a proportionality coefficient determined experimentally

The following condition is accepted for durability:

$$\sigma_i + \int_0^{t_a} M(t - \tau) \sigma_i = \sigma_0 (1 - \gamma c). \quad (3)$$

In addition to the parameters defined above, the following are introduced: σ_i — the stress intensity. The quantities $\sigma_i = [\sigma_r^2 + \sigma_\varphi^2 - \sigma_r \sigma_\varphi]^{1/2}$ included in it, namely σ_r and σ_φ are determined by the previously given formulas (1) and (2). t_a - time to wear (service life until wear).

The case $M(t) = m = \text{const}$ is considered. Using equation (3) and taking into account relations (1) and (2), the time at which the plate begins to lose its functional capability is determined. On this basis, constraints on the magnitude of the applied pressure are established.

The time to corrosion-induced cracking t_* of the plate under the applied pressure is determined using the formula given in [1].

$$t_* = Lt_0(\sigma_i(0, r)) - M.$$

Here, t_0 denotes the time to corrosion-induced cracking under a stationary value of σ_i , i.e., at the initial value of this stress at ($t = 0$) which arises in

the plate in the absence of an aggressive environment. The function t_0 , as well as the constants L and M are characteristics of the “metal-aggressive environment” system; they are determined from appropriate corrosion cracking experiments.

An analytical expression for the fracture front has been obtained as a function of time, the concentration of the aggressive medium, the applied pressure, and the mechanical and geometric parameters of the plate. Furthermore, the time to complete wear of the plate under consideration has been determined.

References

- [1] L.Kh.Talybly, On determining the time to corrosion fracture of metals, Transactions of National Academy of Sciences of Azerbaijan mathematical and mechanics. Baku: "Elm", 2003, vol. XXIII, 1, pp.239-246.
- [2] L. Kh.Talybly, H.A.Mamedova. On a universal function of corrosion strength of materials with regard to the influence of mechanical stresses and the concentration of the diffusing substances .Eastern-European Journal of Enterprise Technologies, 6/11 (72), (2014),46-49.
- [3] Hijran A.Mammadova. On a universal function of long-term corrosive strength // IOSR Journal of Engineering (IOSR JEN) // Vol.11, Issue 2, February 2021, Series I, 58-62

Taking Into Account the Influence of Groundwater on the Durability of a Vertical Well

Mehriban Mammadova

Institute of Mathematics, Baku, Azerbaijan
(e-mail: meri.mammadova@gmail.com)

In the work on the basis of the absent-minded destruction influence of presence of underground waters on change of contact pressure on a joint of the cylindrical non-homogeneous environments surrounding a vertical chink is investigated. The hereditary theory of damageability is put in a research basis. For a regular kernel of damageability curve relaxations of contact pressure of the interlaminar surface, filtrations of underground waters reflecting influence are constructed.

In practice, it is necessary to consider the flow of oil to the oil wells in the calculation of solidity.

The flow to the well bottom of the oil in the layer is complex flow of the filtration. Is the complex problem of learning of the learning of the flow of to be filtering from point of view the hydrodynamic. The causes to be complex filter of the oil its being in the in the different phases and non-homogeneous is of the layer.

In spite of be complex of the flow of to be filtering, to separate the possible to simple flow of the filtering. Because it helps the learning of the complex flows of the filtering. The following conditions are accepted to simple flow of filtration when separates:

- 1) consists of a homogeneous rocks:
- 2) liquid moving in the layer is homogenous:
- 3) filtration flow obeys Darcy's law.

Under the conditions referred to in this one-dimensional, flat radial-flow filtration non-homogeneous environment in the icinity of the bond studied the influence of the well. One-dimensional flow filtration has the following features:

A trajectory of fluids or gas in a porous medium consists of lines that are parallel to each other.

Fluid movement trajectory of the laws of motion are the same for all, by studying the movement of trajectory and therefore can learn all the filtration flow. Such motion of the fluids particle only is expressed with radial

co-ordinate. The perpendicular to the direction of motion of the fluids does not change the pressure.

If motions of the fluids are stable, then pressure only will depend on in all points from the radial co-ordinate. It is necessary to note inform that trajectory of the some fractions falls to on the line of current in the stable motion.

Given the above, the different permeability of concentric circles, which is in contact with the water resistant layer between the contact and the flat-radial flow investigated the effects of stress.

For this environment, consisting of two layers of different permeability is assumed.

The boundaries of the environment will be composed of cylindrical surfaces.

For a regular kernel of damageability curve relaxations of contact pressure of the interlaminar surface, filtrations of underground waters reflecting influence are constructed. The hereditary theory of damageability is put in a research basis.

References

- [1] M.B.Akhundov, S.A.Piriyev, The dispersed failure of a laminated heavy half-plane with a circular aperture. *Transactions of NAS of Azerbaijan*, 2002, 1, 181-185.
- [2] S.A.Piriyev, Forecasting of load - carrying ability of the earth fail around of horizontal cavities. *Global Journal of Geography & Environmental Geo-Sciences*, 2012, 10, .23-25.
- [3] Ju.N.Rabotnov, Introduction in destruction mechanics. Moscow. 1987, 80p.
- [4] N.I.Yaqubov, Calculation well trunk on durability. Moscow. 1982, 183p.

Mathematical Modeling of Oblique Impact of Elastic Waves on a Parallelepiped With Transparent Borders

Ibrakhim Mirzev^{a,b}, Sultan Gaynazarov^c

^a*Tashkent State Transport University, Tashkent, Uzbekistan*

^b*National Research Institute for Seismic Safety and Resilient Construction
under Tashkent Institute Architecture and Civil Engineering, Tashkent,
Uzbekistan*

^c*National University of Uzbekistan, Tashkent, Uzbekistan*
(e-mail: sultangainazarov@gmail.com)

Abstract. When numerically modeling the behavior of underground objects in an infinite soil environment under the influence of seismic waves, it is necessary to limit the area under study; therefore, it is necessary to ensure that both external and internal waves are not reflected at the boundaries. Moreover, the incident wavefront relative to the domain may be oblique, which demands additional transformations for accurate calculation of the impact on the structure. In this work, exact non-reflecting boundary conditions for plane elastic waves with arbitrary incidence angles are presented for the first time. Based on the proposed mathematical model, the behavior of a parallelepiped under waves incident at different angles is studied. The dynamic problem is solved numerically using the finite element method and Bathe's time integration method. The presented results show good quality of non-reflection conditions at artificial boundaries at any angle of wave incidence.

Problem Statement

Consider an infinite half-space in which a parallelepiped is artificially selected. Let a plane elastic wave fall on the parallelepiped, with its wavefront plane rotated by angle α relative to axis OY' and angle β relative to axis OX' in the Cartesian coordinate system $OX'Y'Z'$.

The stress-strain state arising from wave propagation in the domain Ω is determined by the weak form of the equilibrium equation:

$$\int_{\Omega} \lambda \operatorname{div} u \operatorname{div} \delta u + 2\mu \varepsilon^T \delta \varepsilon - (r - \rho \ddot{u} - c\dot{u}) \delta u \, d\Omega - \int_{\Gamma} P \delta u \, d\Gamma = 0 \quad (1)$$

Applying partial discretization with finite elements yields:

$$M\ddot{U} + C\dot{U} + KU = P, \quad U(0) = 0, \quad \dot{U}(0) = 0 \quad (2)$$

where M , C , K are the mass, damping, and stiffness matrices, respectively; U is the vector of nodal displacements; and P is the load vector defined by boundary conditions [1, 2].

Construction of Computational Schemes. A key aspect of calculating the stress-strain state of a bounded domain under wave propagation is defining boundary conditions that ensure transparency for incoming waves and waves reflected from objects inside the domain. Consider a plane wave incident on the lower left corner of the parallelepiped, with particle velocities (v_p, v_s) defined by time functions or digitized earthquake velocigrams:

Particle velocities are defined as:

$$v_p = f_p(t_p), \quad v_s = f_s(t_s) \quad (3)$$

where t_p, t_s is the time representing the difference between the times of appearance of longitudinal and transverse waves at the observation point and the current time. In the calculations, the particle velocity wave function was presented as:

$$f(t) = A_n \cos(2\pi\omega t), \quad A_n = \begin{cases} A \frac{t}{t_n}, & t < t_n \\ A, & t \geq t_n \end{cases} \quad (4)$$

Boundary Conditions

For longitudinal wave incidence on the lower boundary, stresses are expressed as:

$$P_{z'} = -\left(c_p \rho (v_{z'} - v_{pz} \cos \alpha \cos \beta) + \lambda \frac{\partial u_{x'}}{\partial x'} + \frac{\partial u_{y'}}{\partial y'} - \lambda v_{pz} c_p (\sin 2\alpha \cos 2\beta + \sin 2\beta) - c_p \rho v_{pz} \cos 2\alpha \cos 2\beta\right), \quad (5)$$

$$P_{x'} = -\left(c_s \rho (v_{x'} - v_{pz} \sin \alpha) - (\lambda v_{pz} c_p - c_p \rho v_{pz}) \sin \alpha \cos \alpha \cos \beta\right), \quad (6)$$

$$P_{y'} = -\left(c_s \rho (v_{y'} + v_{pz} \cos \alpha \sin \beta) - (\lambda v_{pz} c_p - c_p \rho v_{pz}) \cos 2\alpha \sin \beta \cos \beta\right). \quad (7)$$

Similar conditions are derived for other boundaries.

As an example, Fig. 1 shows the passage of a harmonic longitudinal wave at $\alpha = 0$, $\beta = 30^\circ$ at time $t = 0.7 \text{ sec}$

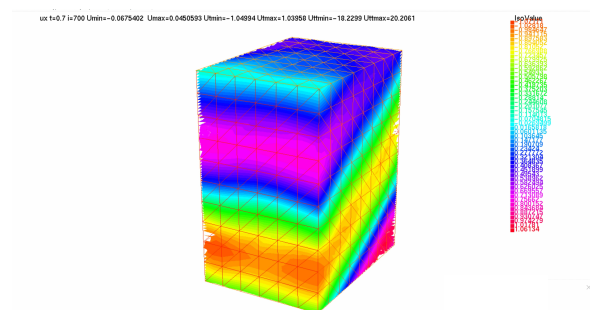


Figure 2: Propagation of a harmonic longitudinal wave

References

- [1] I. Mirzaev, S. Gaynazarov, Impact of Plane Waves on Lining in Elastic Medium With Transparent Artificial Boundary, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.20, pp. 31–40, 2025. <https://doi.org/10.37394/232011.2025.20.4>
- [2] I. Mirzaev, S. Gaynazarov, Numerical modelling of the impact of a plane elastic wave on an artificially selected rectangular domain, *Computational Technologies*, Vol.30, No.4, pp. 54–65, 2025. <https://doi.org/10.25743/ICT.2025.30.4.006>

Three -Dimensional Dynamics of Non-Axisymmetric Moving Load Acting in the Interior of the Bi-Layered Hollow Cylinder

Mahir Mehdiyev^a, Sahib Aliyev^b

^aAzerbaijan State University of Economics (UNEC), Baku, Azerbaijan

^bNakhchivan Institute of Teachers, Nakhchivan, Azerbaijan

(e-mail: mahirmehdiyev@mail.ru, Sahib1960elm@gmail.com)

In the paper within the model of a piecewise-homogeneous body using the exact three-dimensional equations and relations of elastodynamics the dynamics of non-axisymmetric circumferentially distributed and point-concentrated

in the axial direction load moving at a constant velocity V , acting in the interior of a bi-layerd hollow cylinder, is studied (Fig.1).

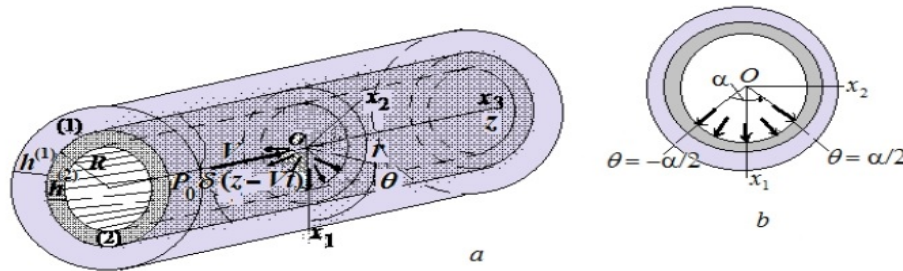


Fig. 1. Sketches of a bi-layerd cylinder (a), of non-axisymmetric distributed loads (b)

This, taking the above into account, a three-dimensional non-axisymmetric dynamic problem of the stress-strain state in a hollow bi-layered cylinder is studied using the exact three-dimensional equations and relations of elastodynamics [3]

The equation of motion and the elasticity relation are as follows [1,2]:

Boundary conditions in the interior of hollow cylinder:

$$\sigma_{rr}^{(2)}|_{r=R-h_2} = \begin{cases} -P_\alpha \delta(z - Vt) & \text{as } -\frac{\alpha}{2} \leq \theta \leq \frac{\alpha}{2}, \\ 0 & \text{as } \theta \in ([-p, p] - (-\frac{\alpha}{2}, \frac{\alpha}{2})), \end{cases}$$

$$\sigma_{r\theta}^{(2)}|_{r=R-h_2} = 0, \sigma_{rz}^{(2)}|_{r=R-h_2} = 0, \quad (1)$$

where P_α in (1) is defined from the following relation:

$$\int_{-\alpha/2}^{+\alpha/2} P_\alpha (R - h) \cos \theta d\theta = (R - h) P_0 = \text{const} \Rightarrow P_\alpha = P_0 / 2 \sin(\alpha/2).$$

Boundary conditions in the interior of hollow cylinder:

$$\sigma_{rr}^{(1)}|_{r=R+h_1} = 0, \sigma_{r\theta}^{(1)}|_{r=R+h_1} = 0, \sigma_{rz}^{(1)}|_{r=R+h_1} = 0. \quad (2)$$

Contact conditions are written in the following form:

$$\begin{aligned}\sigma_{rr}^{(2)}|_{r=R} &= \sigma_{rr}^{(1)}|_{r=R}, \sigma_{r\theta}^{(2)}|_{r=R} = \sigma_{r\theta}^{(1)}|_{r=R}, \\ \sigma_{rz}^{(2)}|_{r=R} &= \sigma_{rz}^{(1)}|_{r=R}, u_r^{(2)}|_{r=R} = u_r^{(1)}|_{r=R}, \\ u_\theta^{(2)}|_{r=R} &= u_\theta^{(1)}|_{r=R}, u_z^{(2)}|_{r=R} = u_z^{(1)}|_{r=R}.\end{aligned}\quad (3)$$

So, investigation of the problem is reduced to the boundary-contact problem (1)-(3), for the solution of which we use the method developed in articles [1-3].

Pass to the moving coordinate system $O'r'z'$, connected to the moving load where the coordinates of points in the system are defined by the following relations:

$$r' = r, z' = z - Vt, \theta' = \theta \quad (4)$$

After transition to a moving coordinate system, the exponential Fourier transform with respect to the coordinate z' is applied to all the above equations and relations. Using corresponding mathematical transforms the analytical expressions of Fourier transforms of desired variables are defined, and their originals are numerically found with the help of algorithm described in [1-3].

As an example, a numerical calculation at $E^{(1)}/E^{(2)} = 0.02$, $\rho^{(1)}/\rho^{(2)} = 0.01$, $\nu^{(1)} = \nu^{(2)} = 0.25$, $c_2^{(1)}/c_2^{(2)} = 1.4142$, is presented, the results of which are given in the Table for various values of the relations h_1/h_2 and h_2/R . When

Calculating for each specifically selected value of parameters of the problem two critically velocities are obtained. The maximum values are given in the numerator, while minimum values are given in the denominator of the fraction presented in the Table. For all values of the relation h_2/R at $h_1/h_2 \rightarrow \infty$ the values of the minimum critical velocity obtained for the system "hollows cylinder + infinite surrounding medium".

Table

The values of dimensionless critical velocity $c_{kp}(= V_{kp}/c_2^{(2)})$ obtained for different values of relations h_2/R and h_1/h_2						
$\frac{h_2}{R}$	h_1/h_2					
	1	2	3	5	7	∞
0.1	0.8710	0.8700	0.8700	0.8700	0.8700	0.8700
	0.3970	0.4200	0.4365	0.4474	0.4494	0.4494
0.2	0.8900	0.8900	0.8900	0.8900	0.8900	0.8900
	0.5467	0.5600	0.5648	0.5662	0.5663	0.5663
0.3	0.9000	0.9000	0.9000	0.9000	0.9000	0.9000
	0.6532	0.6600	0.6613	0.6615	0.6615	0.6615
0.5	0.8038	0.8052	0.8052	0.8052	0.8052	0.8052
	0.4174	0.4384	0.4758	0.5637	0.6122	0.6250

References

- [1] Ozisik, M., Mehdiyev, M.A., Akbarov, S.D. The influence of the imperfectness of contact conditions on the critical velocity of the moving load acting in the interior of the cylinder surrounded with elastic medium *USA: Computers, Materials & Continua*, 54(2), 2018, 103-136.
- [2] Akbarov, S.D., Mehdiyev, M.A., Ozisik, M. Three-dimensional dynamics of the moving load acting on the interior of the hollow cylinder surrounded by the elastic medium. *Seoul: Structural Engineering & Mechanics*, 67(2), 2018, 185-206.
- [3] Akbarov, S.D., Mehdiyev, M.A., Zeynalov A.M. Dynamics of the moving ring-load acting in the interior of the bi-layered hollow cylinder with imperfect contact between the layers. *TWMS J. Pure Appl. Math.* 12(2), 2021, 223-242.

Data-Driven Long-Term Monitoring After Carbon Monoxide Intoxication: A Framework for Clinical Decision Support

Irada Mirzazadeh, Gulchin Abdullayeva

Institute of Mathematics, Baku, Azerbaijan

(e-mail: irada811@gmail.com, gulchinabdullayeva1947@gmail.com)

1. Introduction. CO poisoning is a critical medical condition characterized by nonspecific symptoms and frequent diagnostic delays. Despite advances in acute management, post-acute monitoring remains insufficiently standardized [1,2]. Delayed complications, particularly delayed neurological sequelae (DNS), may occur weeks or months after exposure and significantly impact patient outcomes [3].

In addition to neurological impairment, cardiovascular and renal complications contribute to long-term morbidity [4,5]. The lack of structured follow-up systems highlights the need for interdisciplinary approaches integrating clinical medicine, statistical modelling, and digital technologies.

The aim of this thesis is to develop a data-driven monitoring framework for early detection of complications and individualized risk prediction.

2. Epidemiology and Clinical Significance. CO poisoning is responsible for substantial global mortality and morbidity. According to the Global Burden of Disease Study, approximately 28,900 deaths occurred worldwide in 2021 [1]. Mortality rates are highest in older populations and vary significantly across regions, with Eastern Europe showing the highest burden [1].

In Germany, approximately 3,000–4,000 hospitalizations are recorded annually, with mortality rates reaching up to 15% [6]. In the United States, around 50,000 cases occur annually, with mortality rates between 1

Pathophysiologically, CO poisoning leads not only to hypoxia but also to mitochondrial dysfunction and oxidative stress, resulting in multi-organ damage [4,8].

Major complications include: neurological sequelae (up to 40% of cases) [9,10], cardiovascular events, including myocardial ischemia [11], and renal injury with risk of chronic kidney disease [5]. These findings emphasize the importance of long-term monitoring strategies.

3. Methodology. After discharge, patients undergo structured follow-up based on repeated measurement of clinical parameters, including vital signs, laboratory values, and functional indicators. Data are stored and processed within an electronic health system enabling continuous analysis.

Time series analysis is applied to evaluate longitudinal clinical data and identify trends, seasonal variations, and prognostic patterns. Smoothing techniques such as moving averages and exponential smoothing are used to reduce noise and improve interpretability [12]. Analytical models (linear, exponential, logarithmic) are applied to describe disease progression mathematically.

Regression models are employed to assess relationships between clinical variables and outcomes.

- Simple linear regression is used for single-parameter analysis
- Multiple regression allows multivariate risk modelling

These approaches support individualized prognosis and early detection of adverse developments.

Due to the non-normal distribution of clinical data, non-parametric statistical methods are applied: Mann–Whitney U test, Wilcoxon signed-rank test, Friedman test and Kruskal–Wallis test. These methods ensure robust statistical evaluation of heterogeneous datasets.

An intelligent clinical information system integrates all collected data and provides:

- real-time trend analysis
- automated alerts for critical deviations
- visualization of patient trajectories
- support for clinical decision-making

This system enables a personalized and data-driven monitoring approach.

4. Results and Clinical Implications. The proposed framework improves: 1) early detection of pathological changes 2) risk stratification 3) planning of follow-up care 4) prevention of delayed complications. Particularly vulnerable populations benefit from continuous monitoring and predictive analytics.

5. Discussion. The integration of mathematical modelling and digital health technologies represents a significant advancement in post-CO poisoning care. Unlike conventional approaches, this framework enables dynamic, individualized monitoring.

However, implementation requires:

- integration into clinical IT systems
- validation using longitudinal datasets

- training of medical personnel
- standardization of monitoring protocols

References

- [1] Moberg ME, et al. Global, regional, and national mortality due to unintentional carbon monoxide poisoning, 2000–2021. *Lancet Public Health*. 2023; 8(11): 839–849.
- [2] Mattiuzzi C, Lippi G. Worldwide epidemiology of carbon monoxide poisoning. *Hum Exp Toxicol*. 2020; 39(4): 387-392.
- [3] Kassa BD, et al. Delayed neuropsychiatric sequelae due to carbon monoxide poisoning. *Toxicol Rep*. 2023; 11: 36-39.
- [4] Rose JJ, et al. Carbon monoxide poisoning: Pathogenesis and management. *Am J Respir Crit Care Med*. 2017; 195(5): 596-606.
- [5] Wei KY, et al. Carbon monoxide poisoning and chronic kidney disease risk. *Am J Nephrol*. 2021; 52(4): 292–303.
- [6] S2k Guideline. Diagnosis and treatment of carbon monoxide poisoning. *GMS German Medical Science*. 2021;19:Doc12.
- [7] Shin M, et al. Morbidity and mortality of CO poisoning in the United States. *Ann Emerg Med*. 2022.
- [8] Motterlini R, Foresti R. Biological signaling by carbon monoxide. *Am J Physiol Cell Physiol*. 2017;312:C302–C313.
- [9] Yelmo-Cruz S, et al. Delayed neuropsychiatric syndrome after CO poisoning. *Actas Esp Psiquiatr*. 2022;50(1):65–67.
- [10] Wang T, Zhang Y. Mechanisms and therapeutic targets of CO poisoning. *Chem Biol Interact*. 2024;403:111223.
- [11] Bahng Y, et al. CO poisoning and ischemic heart disease. *Toxics*. 2021;9(10):239.
- [12] Mirzazadeh IH. Monitoring of patients after CO poisoning. *Caspian J Appl Math Ecol Econ*. 2023;10(2):19–42.

Cyclic Torsion of a Viscoelastoplastic Bar: Deformation and Long-term Strength Prediction

Ayten Muradova

Institute of Mathematics, Baku, Azerbaijan
(e-mail: muradovaayten11@gmail.com)

Let the problem of cyclic alternating torsion of a straight circular bar made of viscoelastoplastic material. Using V.V.Moskvitin's constitutive relations with nonlinear function $\varphi^{(1)}(\varepsilon_+^{(1)})$ relaxation kernel $R(t)$, governing equations for twist angle and shear stresses are derived. Two criteria of long-term strength are formulated: damage initiation criterion and cyclic fracture criterion for variable loading. By approximating material time functions as exponentials, the criteria are reduced to analytical relations connecting number of cycles to failure N_* , torque amplitude M_0 , and material constants $m_1, m_2, D, A, B, \alpha, \beta$.

The stress-strain of the circle bar of the viscoelastoplastic material for cyclic torsion is determined. The times of origin of damages in the body and the time before failure are calculated considering the action of stress-state.

References

- [1] V.V. Moskvitin, *Cyclic loading of structural elements*, Nauka, Moscow, 1981, 344p.
- [2] L.Kh. Talybly, M.A. Mamedova, *On accounting for the incubation period of damage accumulation in the low-cycle fatigue criterion for viscoelastoplastic bodies under non-stationary loading*, Proceedings of IMM of Azerbaijan NAS, 1997, vol. VI, 259-266.
- [3] M.A. Mamedova, *Experimental verification of one variant of the low-cycle fatigue criterion for viscoelastoplastic bodies*, Proceedings of Azerbaijan NAS, 1998, vol. XVIII, No. 2, 118-122.

Theoretical Investigation of Elastic-Plastic Cyclic Torsion and Fatigue Features Minimum Cross-Section Rectangular Torque Structural Elements

Nigar Nagiyeva

Institute of Mathematics, Baku, Azerbaijan
Higher Attestation Commission under the President of the Republic of
Azerbaijan, Baku, Azerbaijan
(e-mail: Nigar-sadigova-8@mail.ru)

Abstract. The paper theoretically studies pulsational torsion and fatigue of a narrow rectangular cross-section elastic-plastic bar. Based in energy based methods and hysteresis phenomenon, analytic formulas were derived within elastic-plastic boundaries. The sizes of elasticity and plasticity zones were related with the amplitude and frequency of the torque. The results show that under the elastico-plastic torsion, energy dissipation and fatigue strength is strongly dependent on the loading parameters applied to the bar. The suggested model can be used in practical engineering applications to make calculations for real material patterns.

References

- [1] Smith, J., & Brown, A. Thin rectangular cross-section beam mechanics. Journal of Structural Engineering, 144(9),(2018) 04018112.
- [2] Wang, L., & Zhao, Y. Weight optimization of narrow beams under torsion. Materials Science and Engineering A, 779, (2020) 139120.
- [3] Timoshenko, S., & Goodier, J. N. Theory of Elasticity (3rd ed.). McGraw-Hill.(1970).
- [4] Lemaitre, J., & Chaboche, J. L. Mechanics of Solid Materials. Cambridge University Press.(1990).

Electrokinetic Effects During the Displacement of Miscible Electrolytes in a Porous Medium

Geylani Panahov, Eldar Abbasov, Afat Yuzbashiyeva,
Ibrahim Mamedov

Institute of Mathematics, Baku, Azerbaijan,
(e-mail: eldar.abbasov@gmail.com)

During reservoir development, various waterflooding methods using seawater or formation water are applied to maintain reservoir pressure. The incompatibility of the injected and displaced fluids, which differ in their physicochemical properties, leads to a number of complications accompanied by water breakthrough through highly permeable channels of the oil-bearing formation, a decrease in the injectivity of injection wells, and other phenomena [1 - 4].

Changes in the specific electrical conductivity and density of fluids during displacement in a porous medium were investigated. This phenomenon was observed at different concentration ratios, both at $C_1 > C_2$ and at $C_2 > C_1$, where subscripts 1 and 2 refer, respectively, to the concentrations of the displacing and displaced fluids. At $C_{1,2} = 0 - 2\%$, an increase in the flow rate Q occurred when seawater of lower concentration was displaced by a solution of higher concentration.

When the capillary-osmotic flow ($C_1/C_2 < 1$) is directed opposite to the pressure-driven flow, the flow rate decreases. In the opposite case, ($C_1/C_2 > 1$), this flow coincides in direction with the pressure-driven flow and promotes an earlier breakthrough of the displacing fluid. Studies conducted at different NaCl concentration ratios in the displaced and displacing fluids showed that, at concentrations of 2–4%, a decrease in the flow rate was observed in all directions, accompanied by fluctuations in the flow-rate values over time.

Accordingly, at the first stage, charge accumulation occurs together with diffusion, resulting in the formation of a diffuse layer according to the electrostatic principle. Subsequently, NaCl layers are formed - an adsorption layer, possibly in the form of alternating “hemimicelles”. In the most general form, the flow of two miscible fluids with different physicochemical properties

is described by the following system of differential equations:

$$m \frac{\partial C}{\partial t} + \frac{k}{\mu(C)} [\text{grad} p - g \rho(C) \text{grad} C] - D \Delta C = 0,$$

$$\text{div} \left[\frac{k}{\mu(C)} (\text{grad} p - g \rho(C)) \right] = 0.$$

Here, C is the concentration of the solution at a given point in the porous medium; m is the porosity; k is the permeability; p is the pressure; $\mu(C)$ and $\rho(C)$ are, respectively, the viscosity and density of the solution, which are functions of concentration; and D is the diffusion coefficient. For the one-dimensional case, the equation describing the change in concentration is written as follows:

$$m \frac{\partial C}{\partial t} + \frac{V(t)}{x^n} \frac{\partial C}{\partial x} - D \frac{1}{x^n} \frac{\partial}{\partial x} \left(x^n \frac{\partial C}{\partial x} \right) = 0.$$

The conducted studies show that an increase in the concentration of electrolytes in the solution leads to compression of the diffuse layer. Such regulation can be carried out by selecting the ratio of NaCl concentrations in the displaced and displacing fluids. It was established that, at identical displacement coefficients, the value of the accumulated water factor increases with an increase in the electrical conductivity of the displacing fluid (Fig. 1). Consequently, after the appearance of water in the produced fluid, the efficiency of oil displacement is lower, the greater the difference between the electrical conductivities of the displaced and displacing fluids.

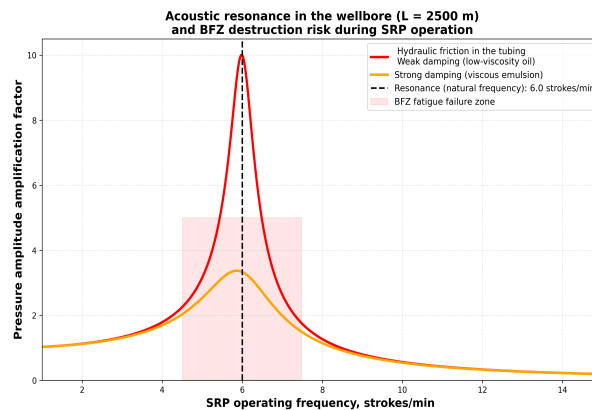


Fig. 1. Dependence of the displacement coefficient on NaCl concentration and the type of displacing fluid

As can be seen from the experimental results, the counterions, carrying the fluid particles along with them, begin to move in the direction opposite to the pressure gradient, which causes retardation of hydrodynamic filtration. The flow rate in each of the layers, taking into account the electroviscous effect, is determined by the formula:

$$Q_i = Q_{0i} - \Delta Q_i = \frac{\Delta P F_i k_i}{l \mu_i} \left(1 - \frac{m_i \varepsilon_i^2 \xi_i^2}{k_i \mu_i \gamma_i} \right), \quad i = 1, 2,$$

where Q_{0i} is the Darcy fluid flow rate; ΔQ_i is the flow rate caused by the electroosmotic flow retarding filtration; ε_i is the dielectric permittivity; ξ_i is the electrokinetic potential; γ_i is the specific electrical conductivity of the fluid. Injection into the reservoir of seawater with electrical conductivity corresponding to the electrical conductivity of the displaced fluid will make it possible to achieve an increase in the injectivity of injection wells, equalization of the profile, and coverage of the reservoir by waterflooding. Such regulation can be carried out by selecting the ratio of NaCl concentrations in the displaced and displacing fluids. When water with an electrolyte additive is injected, due to the elimination of the electroviscous effect, equalization of the filtration profile and an increase in the injectivity of injection wells occur.

References

- [1] M.A. Bagrezaie, B. Dabir, F. Rashidi, et al., *Modeling of formation damage during smart water flooding in sandstone reservoirs*, Scientific Reports, 2023, 13, 17564. <https://doi.org/10.1038/s41598-023-44160-0>.
- [2] J.T. Tetteh, et al. *Predicting the electrokinetic properties on an outcrop and reservoir composite carbonate surfaces in modified salinity brines*, Scientific Reports, 2022, 12. <https://doi.org/10.1038/s41598-022-13892-y>.
- [3]] L.O. Boampong, R. Rafati, A.S. Haddad *Analysis of wettability alteration in low salinity water flooding using a zeta potential-based model*, Capillarity, 2023, 7(2), 32–40. <https://doi.org/10.46690/capi.2023.05.02>.
- [4] A. Kazemi, S. Khezerloo-ye Aghdam, M. Ahmadi *Theoretical and experimental investigation of the impact of oil functional groups on the performance of smart water in clay-rich sandstones*, Scientific Reports, 2024, 14, 20172. <https://doi.org/10.1038/s41598-024-71237-1>.

Assessment of Near-Wellbore Formation Stability Under Hydromechanical and Thermochemical Effects

Geylani Panahov^a, Nazim Valiyev^b, Eldar Abbasov^a,
Babek Sultanov^c

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*SOCAR, 121 Heydar Aliyev Avenue, Baku, Azerbaijan*

^c*Institute of Geology of Azerbaijan, Baku, Azerbaijan*
(e-mail: eldar.abbasov@imm.az)

The stability of the near-wellbore formation zone is one of the key factors determining the reliable operation of oil and gas-condensate wells. Near-wellbore treatment operations may improve fluid inflow while simultaneously initiating the failure of cementing bonds, sand production, and cavity formation [1 - 3].

The aim of this study is to provide an integrated assessment of the conditions leading to near-wellbore formation failure under the combined effects of rock-skeleton consolidation, thermochemical processes during acid treatment, CO₂ evolution, pressure-pulse propagation, and production-induced drawdown. A decrease in reservoir pressure during field development leads to an increase in effective stresses and consolidation of the reservoir skeleton. The Terzaghi–Biot law was used as the basic estimate:

$$\sigma_{eff} = \sigma_{total} - \alpha \cdot P_{pore}, \quad (1)$$

where σ_{total} - total geostatic stress (lithostatic pressure of the overlying rocks), P_{pore} - pore pressure, α - the Biot coefficient characterizing the relationship between pore pressure and effective stress. According to an estimate based on the Kozeny–Carman model, even a moderate decrease in porosity may lead to a decrease in permeability by approximately 14%, which leads to a significant deterioration in well productivity. During acid treatment of carbonate inclusions and the cementing material of sandstone, the reaction is accompanied by the release of heat and gaseous CO₂ directly in the reservoir pores. The intensity of gas generation is determined by the reaction rate constant. To account for the influence of temperature and pressure, the modified Arrhenius equation may be used:

$$k(T, P) = A \cdot e^{\frac{\Delta V \cdot P - E_a}{R \cdot T}}. \quad (2)$$

The calculation estimates show that an increase in temperature by 30°C and pressure by 10 MPa can accelerate the reaction by approximately 4 times. This significantly increases the intensity of local gas release and the pore-pressure impulse. In this case, the thermal effect is considered not as an independent mechanism of formation failure, but as a factor that reduces rock strength and intensifies the effect of drawdown and impulse pressure. The pressure impulse arising during gas release is described by a wave equation with attenuation in a saturated porous medium:

$$\frac{\partial^2 P}{\partial t^2} + 2\gamma \frac{\partial P}{\partial t} = c^2 \nabla^2 P + S(r, t); L_{\text{decay}} = \frac{c}{\gamma}, \quad (3)$$

where: P - formation pressure, Pa; c - speed of sound in a saturated porous medium, m/s; γ - attenuation coefficient, s^{-1} ; $S(r, t)$ - volumetric pressure source due to gas generation, Pa/s²; L_{decay} - attenuation length, i.e., the distance over which the amplitude decreases by a factor of $e \approx 2, 7$.

For a high-frequency impulse, the calculation shows extremely rapid attenuation: in a plane formulation, the characteristic length may amount to fractions of a millimeter. In a spherical formulation, geometric attenuation is additionally manifested, and therefore the impulse amplitude decreases even more rapidly.

The most significant difference between an oil formation and a gas-condensate formation is associated with the pressure diffusivity coefficient. In an oil formation, $k = 200$ mD; $\varphi = 0.20$; $\mu = 5 \cdot 10^{-3} \text{ Pa} \cdot \text{s}$ and $\beta = 10^{-9} \text{ Pa}^{-1}$, the calculated pressure diffusivity is about $0.2 \text{ m}^2/\text{s}$, and the response time at a distance of 100 m is approximately 3.5 h. In a gas-condensate formation, due to the lower viscosity and higher compressibility of gas, the pressure diffusivity increases to $\sim 0.33 \text{ m}^2/\text{s}$, and the response time decreases to ~ 2.08 h. At the same well depth and comparable reservoir pressures, in a gas-condensate formation, a decrease in wellhead pressure leads to a sharper increase in drawdown in the near-wellbore formation zone. An additional risk factor is the acoustic resonance of the fluid column in the well during the operation of a sucker-rod pump (Fig. 1).

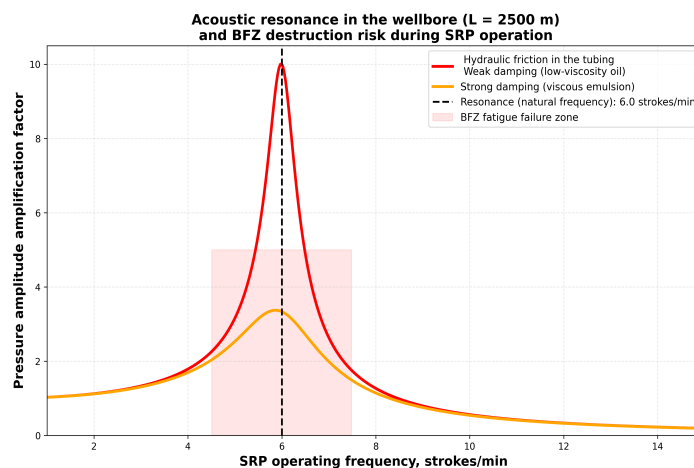


Fig. 1. Resonance effect on the failure of the near-wellbore formation zone

As a result of the conducted studies, it has been shown that the failure of the near-wellbore formation zone is the result not of a single factor, but of the combined action of consolidation, thermochemical weakening, gas generation, pressure-diffusion redistribution, and production drawdown. The developed approach makes it possible to consider geomechanical, thermochemical, and hydrodynamic factors from a unified point of view in the process of assessing the stability of the near-wellbore formation zone. The operating conditions requiring a special approach are processes in which acid treatment and gas generation coincide with high drawdown, rapid well start-up, or resonant oscillations of the fluid column.

References

- [1] Ahmed, T., *Reservoir Engineering Handbook*, 5th ed. Gulf Professional Publishing, 2019.
- [2] Civan, F., *Reservoir Formation Damage: Fundamentals, Modeling, Assessment, and Mitigation*. 3rd ed. Gulf Professional Publishing, 2015.
- [3] Panahov, G., Museibli, P., Sultanov, B., *Effect of soil consolidation on the fractality of the filtration law*, International Journal of Applied Mechanics and Engineering, 28(1), 84–94. <https://doi.org/10.59441/ijame-2023-0008>.

Solution of Lamb's Problem for an Elastic Half-Space

Nazila Rassoulova^a, Mubariz Rassoulov^a, Tehmine Mahmudzade^b

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Baku State University, Baku, Azerbaijan*

(e-mail: rasulova@gmail.com, tehminemahmudzade1996@gmail.com)

Lamb's problem continues to attract considerable attention among researchers in Solid Mechanics due to its fundamental significance in elastodynamics and seismic wave propagation. Originally formulated by Lamb in 1904, this problem models earthquake-induced wave propagation in the Earth's crust for both two-dimensional and three-dimensional configurations.

The first exact analytical solution to the plane Lamb problem was obtained by V.V. Poruchikov [1]. However, this solution is mathematically complicated, requiring the inversion of double integral transforms (Laplace and Fourier transforms) using the Cagniard method.

In [2], an alternative and more efficient analytical approach was proposed. This method led not only to a simpler solution of the same problem but also to the discovery of a completely new class of homogeneous solutions of plane elastodynamic problems, as well as a new type of Smirnov–Sobolev functionally invariant solutions.

In the present work, a similar analytical approach is employed to solve Lamb's problem for an elastic half-space. Hankel and Laplace transforms are applied, and the corresponding boundary functions of the unknown solutions are determined in the transform domain.

The equations of motion are given by

$$\begin{aligned}\rho \frac{\partial^2 u_r}{\partial t^2} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} \\ \rho \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \frac{\partial \sigma_{rz}}{\partial r} \\ \frac{\partial^2 \varphi}{\partial t^2} &= c_1^2 \nabla^2 \varphi ; \quad \frac{\partial^2 \psi}{\partial t^2} = c_2^2 \nabla^2 \psi \\ \frac{d^2 \tilde{\varphi}}{dz^2} - \left(k^2 + \frac{s^2}{c_1^2} \right) \tilde{\varphi} &= 0\end{aligned}$$

$$\frac{d^2\tilde{\psi}}{dz^2} - \left(k^2 + \frac{s^2}{c_1^2}\right)\tilde{\psi} = 0$$

After applying the Laplace transform with respect to time and the Hankel transform with respect to the radial coordinate, the governing equations become:

$$\begin{aligned}\tilde{u}_r &= -k\tilde{\varphi} - \frac{\partial\tilde{\psi}}{\partial z} \quad ; \quad \tilde{u}_z = \frac{\partial\tilde{\varphi}}{\partial z} - k\tilde{\psi} \\ \tilde{u}_r &= -kAe^{-az} + \beta Be^{-\beta z} \\ \tilde{u}_z &= -\alpha Ae^{-\alpha z} - kBe^{-\beta z}\end{aligned}$$

$$\begin{aligned}\tilde{u}_r &= -\frac{\alpha I_0 (k^2 + \beta^2) e^{-az}}{[(\lambda + 2\mu) a^2 - \lambda k^2] (k^2 + \beta^2) + 4k^2 a \beta} + \frac{2I_0 k a \beta e^{-\beta z}}{[(\lambda + 2\mu) a^2 - \lambda k^2] (k^2 + \beta^2) + 4k^2 a \beta} \\ \tilde{u}_z &= -\frac{a I_0 (k^2 + \beta^2) e^{-az}}{[(\lambda + 2\mu) a^2 - \lambda k^2] (k^2 + \beta^2) + 4k^2 a \beta} - \frac{2I_0 k^2 a e^{-\beta z}}{[(\lambda + 2\mu) a^2 - \lambda k^2] (k^2 + \beta^2) + 4k^2 a \beta}\end{aligned}$$

Each term of this formula can be represented in the following form:

$$\tilde{u}_m = \frac{I \cdot c_m}{\mu} \cdot \frac{e^{z \cdot \sqrt{k^2 + \frac{s^2}{c_m^2}}}}{\sqrt{c_m^2 \cdot k^2 + s^2}} \cdot \sqrt{1 + \frac{s^2}{k^2 c_m^2}} \cdot f\left(\sqrt{1 + \frac{s^2}{k^2 c_m^2}}\right) \quad m = 1, 2$$

This makes it possible to apply the method proposed in (2), which enabled the derivation of an effective solution to Lamb's problem for a half-plane. Although the problem in our case is three-dimensional, we have nevertheless succeeded in constructing a solution for it as well.

References

- [1] Poruchikov, V. B. *Methods of Classical Theory of Elastodynamics*. Berlin/Heidelberg: Springer-Verlag, 1993. 350 p.
- [2] Rassoulova, N. B., and Rassoulov, M. B. A New Class of Homogeneous Solutions for Plane Elastodynamic Problems. *Mechanics of Solids*, vol. 55, no. 7, 2020, pp. 1057–1061.

Vibrations of a Cylindrical Shell Stiffened With Inhomogeneous Rods and Rings

Vusala Rzayeva

Institute of Mathematics, Baku, Azerbaijan
(e-mail: rzayevavusale03@gmail.com)

In connection with the demands of technology, studying the dynamic behavior of structural elements taking into account their adequate material properties is of great importance. Modern mechanical engineering very often poses problems of calculating thin-walled structures with mutually exclusive properties: on one hand, the structures under study must combine high strength and on the other hand, efficiency and lightness. To successfully combine the above properties, it seems entirely justified to use orthotropic materials and plastics in structures.

Depending on mechanical and thermal processing, the type of technology, composition of the material, the homogeneity and anisotropy features are created in the material of the structure. On the other hand, such constructions are in contact with different nature media. In many cases, there arises a need to stiffen such thin-walled structures for increasing their serviceability. The nonhomogeneity of the cylindrical shell change by the linear law in the direction of the thickness. The fluid was accepted as ideal. Rigid contact condition between the rods and the cylindrical shell was considered. Using the contact conditions, the frequency equation was structured, the roots were found implemented by the numerical method, characteristical curves were built. The conducted analyses show that vibrations of a nonhomogeneous cylindrical shell dynamically contacting with moving fluid and stiffened with inhomogeneous rods have not been studied. In the present paper, we consider free vibrations of an inhomogeneous anisotropic, fluid-contacting cylindrical shell stiffened with inhomogeneous rods and rings. The Hamilton-Ostogradsky variational principle was used when solving the problem. It was accepted that the nonhomogeneity of rods and rings used in the strengthening change by the exponential law. The nonhomogeneity of the cylindrical shell change by the linear law in the direction of the thickness. The fluid was accepted as ideal. Rigid contact condition between the rods and the cylindrical shell was considered. Using the contact conditions, the frequency equation was structured, the roots

were found implemented by the numerical method, characteristic curves were built.

Under the cylindrical shell strengthened with rods and rings we understand a cylindrical shell and a system consisting of rods and rings rigidly strengthened to it along the coordinate lines. It is considered that the coordinate axes coincide with the principal curvature lines of the cylindrical shell, rods and rings and are in rigid contact along these lines.

Stability of the Plane Form of Bending of Thin-Walled Rods with Regard to Bimodularity

Tebriz Shamiev, Ruhiyye Zamanova, Rena Ismailova

Azerbaijan Technical University, Baku, Azerbaijan

(e-mail: tebriz.shamiyev@aztu.edu.az, ruhiyye.zamanova@aztu.edu.az,)

Let a rod whose cross-section has one axis of symmetry be subjected to pure bending by couples applied at its ends; moreover, the bending occurs in the plane of symmetry of the rod.

Coordinate axes are chosen as follows. oz axis coincides with the central line of the undeformed rod. x_1 and x_2 axes lie in the plane of the end cross-section. Suppose that the rod is fixed in such a way that the end cross-sections can freely rotate about x_1 and x_2 axes, but can't rotate about z axis, i.e. we have following boundary conditions

$$\begin{aligned} w &= 0 \quad \text{or} \quad z = 0, \quad z = l \\ w' &= 0 \quad \text{or} \quad z = 0, \quad z = l \end{aligned} \tag{1}$$

where w is deflection.

Assuming that in the precritical state the plane-sections hypothesis holds, we can write [1-3],

$$\varepsilon_{21} = \varepsilon_{23} = \varepsilon_{13} = 0, \quad \varepsilon_{11} = \varepsilon_{22} = -\frac{1}{2}\varepsilon_{33}, \quad \varepsilon_{33} = \varepsilon_0 - \kappa_0, \quad x_1 \varepsilon_n = |\varepsilon_{33}|.$$

Here ε_0 is relative deformation, $\kappa_0 = \frac{d^2 w}{dz^2}$ is curvature of the central line before the loss of stability.

In the precritical state across the cross-section of the rod, in general, the following deformation domains will exist: the domains of elastic and plastic deformations due to tension S_y^+ and S_p^- and the domains of elastic and plastic deformations due to compression S_y^- and S_p^- .

Using equilibrium conditions

$$\iint_{\delta} \sigma_{33} dx_1 dx_2 = 0, \quad \iint_S x_1 \sigma_{33} dx_1 dx_2 = M \quad (2)$$

and stress continuity conditions

$$\begin{aligned} \sigma_y^+ \Big|_{x_2=\eta_2} &= \sigma_p^+ \Big|_{x_2=\eta_2} = \sigma_S^+ \quad \text{as } \sigma > 0 \\ \sigma_y^- \Big|_{x_2=\eta_1} &= \sigma_p^- \Big|_{x_2=\eta_1} = \sigma_S^- \quad \text{as } \sigma < 0 \end{aligned} \quad (3)$$

we can define ε_0, χ_0 , and also unknown boundaries of elastic and plastic deformations η_1 and η_2 . Here σ_S^+ and σ_S^- are values of σ_S as $\sigma > 0$, $\sigma < 0$ respectively.

In the precritical state the following dependence between stresses and deformations holds:

$$\sigma_{33} = E\varepsilon_{33} [1 - \omega(\varepsilon_u)] \quad (4)$$

Here $\omega(\varepsilon_u)$ is Il'yushin plasticity function which is denoted by $\omega^+(\varepsilon_u)$, $\omega^-(\varepsilon_u)$ as $\sigma > 0$, $\sigma < 0$ respectively.

In the domains of elastic deformations it should be assumed in relation (4) that $\omega(\varepsilon_u) = 0$.

In cases where the bending stiffness of the rod relative to the x_1 axis is small compared to the stiffness relative to the x_2 -axis, for some critical value of bending moment M_{kp} the plane form of bending becomes unstable.

The perturbed stress state of the rod is characterized by the following stresses:

$$\sigma_{33} + \delta\sigma_{33}, \delta\sigma_{23}, \delta\sigma_{13}, \delta\sigma_u = \delta|\sigma_{33}|$$

and infinitesimal deformed state in the cases of incompressible materials is characterized by the following deformations:

$$\varepsilon_{33} + \delta\varepsilon_{33}, \delta\varepsilon_{23}, \varepsilon_{22} + \delta\varepsilon_{22} = \varepsilon_{11} + \delta\varepsilon_{11} = -\frac{1}{2}(\varepsilon_{33} + \delta\varepsilon_{33}), \delta\varepsilon_{21} = 0.$$

In the loading domains the connection between variations of deformations and stresses can be obtained by varying the fundamental law of small elastic-plastic deformations:

$$\begin{aligned}\delta\sigma_{33} &= E\delta\varepsilon_{33} \left[1 - w(\varepsilon_u) \frac{dw}{d\varepsilon_u} \right] \\ \delta\sigma_{23} &= \frac{E}{3}\delta\varepsilon_{23} [1 - w(\varepsilon_u)] \\ \delta\sigma_{13} &= \frac{E}{3}\delta\varepsilon_{13} [1 - w(\varepsilon_u)]\end{aligned}\quad (5)$$

For domains of elastic deformations and elastic unloading the dependence between variation of stresses and deformations takes the form:

$$\delta\sigma_{33} = E\delta\varepsilon_{33}, \quad \delta\sigma_{23} = \frac{E}{3}\delta\varepsilon_{23}, \quad \delta\sigma_{13} = \frac{E}{3}\delta\varepsilon_{13}$$

The boundary between the loading and unloading domains is determined from the condition that the variations of the deformation intensity are equal to zero:

$$\delta\varepsilon_u = \delta|\varepsilon_{33}| = 0.$$

It is known that the equations for the loss of stability of the plane of bending within elastic limits have the form [6]

$$B_0 \frac{d^2 u}{dz^2} = \psi M; \quad C_0 \frac{d\psi}{dz} = -\frac{du}{dz} M$$

where B_0 is a stiffness at elastic bending, C_0 is an elastic stiffness at torsion, and ψ is an angle of relation of cross-section around z_0 -axis.

Thus the formulated problem amounts to solving the system of two ordinary differential equations

$$\begin{cases} B^0 B^* \frac{d^2 u}{dz^2} = \psi M \\ C^0 C^* \frac{d\psi}{dz} = -M \frac{du}{dz} \end{cases} \quad (6)$$

where B^0 is a stiffness at elastic bending, C^0 is a stiffness at elastic torsion, for bimodular materials, B^* and C^* are constant values.

From (6) we obtain

$$M_{kp} = M_{kp}^0 \sqrt{B^* C^*}, \quad l_{kp} = l_{kp}^0 \sqrt{B^* C^*}$$

where M_{kp}^0, l_{kp}^0 are respectively critical moment and critical length of the rod at which it loses stability while remaining elastic.

References

- [1] V.D. Gadjiev, stability of plates and rods with different resistance to tension and compression, *Mater. Respubl. Cont., molod.uch., Baku, Elm*, (1975), 207-210.
- [2] V.Z. Vlasov, *Thin-walled rods*. M.1975, 507 p.
- [3] V.D. Gadjiev, T.M. Shamiyev, Buskling of viscoelastic rods with different resistance to tension and compression. *Vsesoyuz. Simp. Creeping in constructions. Dnepropetrovsk, DGU*, 1982, p.30.

Influence of Effect of a Sliding Friction on the Characteristic of Natural Frequencies of the Longitudinally Supported Cylindrical Envelopes With Filler at an Axial Compression

Famil Seyfullaev^{a,b}, Alikhan Garayev^b

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
(e-mail: Familseyfullayev2016@gmail.com, alikhan.karayev60@gmail.com)

This thesis is devoted to the study of free vibrations of cylindrical shells with a filler reinforced by discretely distributed longitudinal systems of ribs under axial compression and taking into account friction between the shell and the filler. The analysis of the influence of environmental parameters on the frequency parameters of the natural oscillations of the system.

The problem is solved in an energetic way. The potential energy of the shell loaded with axial compressive forces has the form [1]:

$$\mathcal{E} = \frac{h}{2(1-\nu^2)} \int_0^{\xi_1} \int_0^{2\pi} \left\{ \left(\frac{\partial u}{\partial \xi} + \frac{\partial \nu}{\partial \theta} - w \right)^2 + 2(1-\nu) \frac{\partial u}{\partial \xi} \left(\frac{\partial \nu}{\partial \theta} - w \right) - \right. \\ \left. - \frac{1}{4} \left(\frac{\partial u}{\partial \theta} + \frac{\partial \nu}{\partial \xi} \right)^2 \right\} d\xi d\theta + \frac{h}{24(1-\nu^2)R^2} \int_0^{\xi_1} \int_0^{2\pi} \left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial \nu}{\partial \theta} \right)^2 -$$

$$\begin{aligned}
& -2(1-\nu) \left[\frac{\partial^2 w}{\partial \xi^2} \left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial \nu}{\partial \theta} \right) - \frac{1}{4} \left(\frac{\partial^2 w}{\partial \xi \partial \theta} + \frac{\partial \nu}{\partial \xi} \right)^2 \right] d\xi d\theta + \\
& + \frac{E_c}{2R} \sum_{l=1}^k \int_0^{\xi_1} \left[F_c \left(\frac{\partial u}{\partial \xi} - \frac{h_c}{R} \frac{\partial^2 w}{\partial \xi^2} \right)^2 + \frac{I_{yc}}{R^2} \left(\frac{\partial^2 w}{\partial \xi^2} \right)^2 + \frac{G_c}{E_c} I_{kp.c} \times \right. \\
& \quad \left. \times \frac{G_c}{E_c} I_{kp.c} \left(\frac{\partial^2 w}{\partial \xi \partial \theta} + \frac{\partial \nu}{\partial \xi} \right)^2 \right]_{\theta=\theta_1} d\xi - \\
& - \frac{\sigma_x h}{2} \int_0^{\xi_1} \int_0^{2\pi} \left(\frac{\partial w}{\partial \xi} \right)^2 d\xi d\theta - \frac{\sigma_x F_c}{2R} \sum_{i=1}^k \int_0^{\xi_1} \left(\frac{\partial w}{\partial \xi} \right)^2 \Big|_{\theta=\theta_1} d\xi \quad (1)
\end{aligned}$$

Here $\xi_1 = \frac{L}{R}$, $\xi = \frac{x}{R}$, $\theta = \frac{y}{R}$; x, y, z coordinates, E_c, G_c - moduli of elasticity and shear of the material of the longitudinal ribs, k number of longitudinal ribs, σ_x - axial compressive stresses, u, ν, w - components of the shell displacement vector, h and R - shell thickness and radius, respectively, E, ν - Young's modulus and Poisson's ratio of the shell material, $F_c, I_{yc}, I_{kp.c}$ - respectively, the area and moments of inertia of the cross section of the longitudinal rod relative to the axis OX and OZ, as well as the moment of inertia during torsion.

The kinetic energy of the shell is as follows:

$$\begin{aligned}
K = & \frac{h}{2(1-\nu^2)} \int_0^{\xi_1} \int_0^{2\pi} \left[\left(\frac{\partial u}{\partial t_1} \right)^2 + \left(\frac{\partial \nu}{\partial t_1} \right)^2 + \left(\frac{\partial w}{\partial t_1} \right)^2 \right] d\xi d\theta + \\
& + \frac{\overline{\rho_c} E_c F_c}{2R(1-\nu^2)} \sum_{i=1}^{k_1} \int_0^{\xi_1} \left[\left(\frac{\partial u}{\partial t_1} \right)^2 + \left(\frac{\partial w}{\partial t_1} \right)^2 \right]_{\theta=\theta_1} d\xi \quad (2)
\end{aligned}$$

Here $\overline{\rho_c} = \frac{\rho_c}{\rho_0}$, where ρ_0, ρ_c - the density of the shell materials and the longitudinal rod, respectively, $\theta_i = \frac{2\pi}{k_1} i$.

The interaction of the filler with the shell is represented as a surface load applied to the shell, which performs work on the displacements of the contact surface when the system is transferred from the deformed state to the initial undeformed state.

$$A_0 = - \int_0^{\xi_1} \int_0^{2\pi} (q_x u + q_\theta \nu + q_z w) d\xi d\theta + \int_0^{\xi_1} \int_0^{2\pi} f q_z (u + \nu) d\xi d\theta \quad (3)$$

where q_x, q_θ, q_z - pressure from the filler side to the shell, f - coefficient of friction.

Where $a_t^2 = (\lambda + 2\mu)/\rho, a_e^2 = \mu/\rho$, and a_t, a_e the propagation velocity of longitudinal and transverse waves in the aggregate, respectively; $S = S(S_x, S_\theta, S_z)$ - displacement vector; λ, μ - Lamé coefficients. It is assumed that the contact between the shell and the aggregate is rigid, i.e. at $r = R$:

$$u = S_x; \quad \nu = S_\theta; \quad w = S_z \quad (4)$$

$$q_x = -\sigma_{rx}, \quad q_y = -\sigma_{r\theta}, \quad q_z = -\sigma_{rr}, \quad w = S_r \quad (5)$$

Components $\sigma_{rx}, \sigma_{r\theta}, \sigma_{rr}$ - stress tensors are defined as follows [2,3,4]:

$$\begin{aligned} \sigma_{rx} &= \mu_s \left(\frac{\partial S_x}{\partial r} + \frac{\partial S_r}{\partial x} \right); \quad \sigma_{r\theta} = \mu_s \left[r \frac{\partial}{\partial r} \left(\frac{S_r}{r} \right) + \frac{1}{r} \frac{\partial S_r}{\partial \theta} \right], \\ \sigma_{rr} &= \lambda_s \left(\frac{\partial S_r}{\partial x} + r \frac{\partial}{\partial r} \left(\frac{S_r}{r} \right) + \frac{1}{r} \frac{\partial S_\theta}{\partial \theta} \right) + 2\mu_s \frac{\partial S_r}{r} \end{aligned} \quad (6)$$

λ_s, μ_s - Lamé coefficients for the medium.

As noted, the methodology for determining the optimal reinforcement parameters is based on comparing the minimum vibration frequencies of the ribbed and smooth cylindrical shells reinforced with longitudinal systems of ribs filled with medium [5].

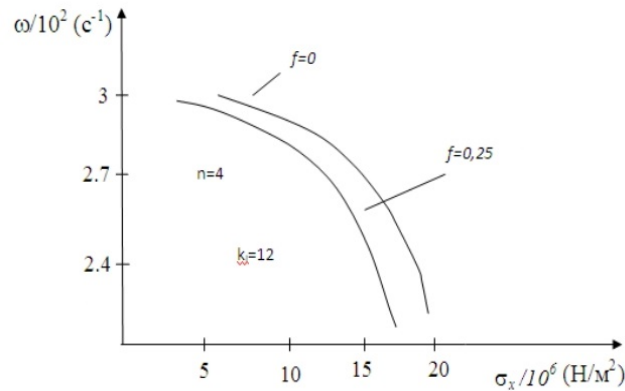


Fig.1. System frequency dependencies $\omega = \omega_1 \omega_0$ against compressive stress

References

- [1] F.S. Latifov, F.A. Seyfullaev, Sh. Sh. Alyev. Free vibrations reinforced by transverse ribs of an anisotropic cylindrical shell made of fiberglass with a liquid flowing in it. Applied mechanics and technical physics. 57(4), 2016, 158-162.
- [2] A.I. Seyfullayev, K. A. Novruzova, Oscillations of longitudinally reinforced orthotropic cylindrical shell filled with a viscous fluid, Eastern-European Journal of Enterprise Technologies, no. 3/7 (75), (2015), 29-33.

NONLINEAR PROBLEMS OF MATHEMATICAL PHYSICS**A Promenade in the Nonlinear World****Attila Aşkar***Koç University, Istanbul, Türkiye*

(e-mail: aaskar@ku.edu.tr)

The paper touches on four important areas of Nonlinear Phenomena as: Chaos, Solitons, Schrödinger equation in the Nonlinear QFD format and the Large deformations of the Helix. Each of these topics is illustrated by a solution and the summary of the ideas that lead to the discovery of each. Chaos is illustrated with the solution of the Henon – Heiles equation; the solution of the Korteweg - de Vries equation is chosen to display the characteristics of Solitons; Schrödinger equation in the Nonlinear Quantum Fluid Dynamics (QFD) format investigates the possibilities of dephasing, dissipation, vorticity and turbulence; the least analytically studied Helix's large displacement dynamics is presented with numerical results along with possibilities for solitons.

**Absence of Global Solutions in an Infinite Domain for a
System of Second-Order Semilinear Equations With
Fractional Time Derivatives**
Shirmail Bagirov*Baku State University, Baku, Azerbaijan**Institute of Mathematics, Baku, Azerbaijan*

(e-mail: sh_bagirov@yahoo.com)

Let $R > 0$, $B_R = \{x : |x| < R\}$, $B'_R = \{x : |x| > R\}$, $\partial B_R = \{x : |x| = R\}$,

$$\bar{B}'_R = R^n \setminus B_R, Q_R = B_R \times (0; +\infty), Q'_R = B'_R \times (0; +\infty),$$

$$\bar{Q}'_R = \bar{B}'_R \times [0; +\infty), x = (x_1, \dots, x_n) \in R^n, r = |x| = \sqrt{x_1^2 + \dots + x_n^2}.$$

In the domain Q'_R we consider the following system of equations:

$$\begin{cases} ({}^C D_{0+,t}^\alpha u_1)(x, t) = \Delta u_1 + \frac{C_1}{|x|^2} u_1 + |x|^{\sigma_1} |u_2|^{q_1} \\ ({}^C D_{0+,t}^\alpha u_2)(x, t) = \Delta u_2 + \frac{C_2}{|x|^2} u_2 + |x|^{\sigma_2} |u_1|^{q_2}, \end{cases} \quad (1)$$

where $n > 2$, $C_i < \left(\frac{n-2}{2}\right)^2$, $\sigma_i \in R$, $q_i > 1$, $i = 1, 2$.

Here

$$({}^C D_{0+,t}^\alpha u)(x, t) = \frac{1}{(1-\alpha)} \int_0^t \frac{u_s(x, s)}{(t-s)^\alpha} ds,$$

$$({}^C D_{T-,t}^\alpha u)(x, t) = -\frac{1}{(1-\alpha)} \int_t^T \frac{u_s(x, s)}{(s-t)^\alpha} ds$$

are the left-sided and the right-sided Caputo fractional derivatives of order α , $u_s(x, s) = \frac{\partial u(x, s)}{\partial s}$, $(1-\alpha)$ is the Gamma function.

We will investigate the issue of the absence of global solutions to the system (1), satisfying the initial condition

$$u_i|_{t=0} = u_{0i}(x) \geq 0, \quad x \in B'_R, \quad (2)$$

$$u_{0i}(x) \geq 0, \quad u_{0i} \in L_{\infty, loc} \in (\bar{B}'_R).$$

A solution is understood in the weak sense, namely a pair of functions $u_1(x, t), u_2(x, t) \in L_{\infty, loc}(\bar{Q}'_R)$ is called a weak solution of problem (1),(2) if

$$\begin{aligned} & \int_{B'_R} \int_0^T |x|^{\sigma_i} |u_j|^{q_i} \eta dx dt = \\ & = \int_{B'_R} \int_0^T u_i (D_{T-,t}^\alpha \eta)(x, t) dx dt - \int_{B'_R} \int_0^T u_i \left[\Delta \eta + \frac{C_i}{|x|^2} \eta \right] dx dt - \\ & - \int_{B'_R} u_{0i}(x) \frac{1}{(1-\alpha)} \int_0^T \frac{\eta(x, t)}{t^\alpha} dt dx + \int_0^\infty \int_{\partial B'_R} u_i \frac{\partial \eta}{\partial \nu} ds dt \end{aligned}$$

for any $\eta(x, t) \in C^{2,1}(\bar{Q}'_R)$, $\eta|_{\partial B_R} = 0$, $\frac{\partial \eta}{\partial \nu}|_{\partial B_R} \leq 0$, and there exist $R_1 > R$, $T > 0$ such that $\eta(x, t) \equiv 0$ when $|x| > R_1$ and $t > T_0$, where ν is the unit vector of the outer normal to ∂B_R , $i, j = 1, 2$, $i \neq j$.

The problems of the absence of global solutions for various classes of differential equations and inequalities play an important role in theory and applications. Therefore, they are under constant attention from mathematicians, and many works have been dedicated to them. An overview of such results can be found in the monograph [1] and in the survey paper [2].

Let us introduce the following notations:

$$\lambda_i^\pm = -\frac{n-2}{2} \pm \sqrt{D_i}, \quad D_i = \left(\frac{n-2}{2}\right)^2 - C_i,$$

$$\theta_i = \frac{\sigma_i + 2 + q_i(\sigma_j + 2)}{q_1 q_2 - 1} - \lambda_i^+ - n - \frac{2(1-\alpha)}{\alpha},$$

where $i, j = 1, 2$, $i \neq j$.

Our main result is the following theorem:

Theorem. *Let $n \geq 3$, $q_i > 1$, $0 < \alpha < 1$, $C_i < \left(\frac{n-2}{2}\right)^2$, $\sigma_i \in R$, $u_{0i}(x) \geq 0$, $u_{0i} \in L_{\infty,loc}(\bar{B}'_R)$, $i = 1, 2$. If $\max(\theta_1, \theta_2) > 0$, then problem (1),(2) has no non-negative solutions in Q'_R .*

To prove this result, we use the test function method developed by Mitidieri and Pokhozhaev in their work [1].

References

- [1] Mitidieri E. and Pokhozhaev S. I., A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, Tr. Mat. Inst. Steklova, 234(2001), 3–383 (Russian) Math-Net.Ru Math-Net.Ru MathSciNet ; English translation: Proc. Steklov Inst. Math., 234 (2001), 1–362
- [2] Deng K. Levine H. A., The role of critical exponents in blow-up theorems: the sequel, J. Math. Anal. Appl., 243 (2000), 85–126.

On Some Liouville Theorems for p -Laplace Type Operators

Michel Chipot

*Institute for Mathematics University of Zurich Winterthurerstrasse 190
CH-8057 Zurich*

(e-mail: m.m.chipot@math.uzh.ch)

The goal of this talk is to consider Liouville type theorem for p -Laplacian type operators. In particular guided by the Laplacian case one establishes analogous results for the p -Laplacian and operators of this type.

Stabilization of Damped Nonlinear Wave Equations by Controllers Depending on Finitely Many Parameters

Varga Kalantarov

Department of Mathematics, Koç University, Istanbul, Türkiye

(e-mail: VKalantarov@ku.edu.tr)

We study the problem of stabilization of solutions of various classes of damped nonlinear wave equations employing feedback control terms that involve parameters, such as, finitely many Fourier modes, finitely many volume elements and finitely many nodal observables and controllers. Specifically we discuss the problem of stabilization to a given steady state solution of initial boundary value problems for nonlinear weakly damped wave equation, nonlinear wave equation with structural damping, nonlinear wave equations with nonlinear damping term (including marine riser equation) and some related nonlinear wave equations.

On the Weak Solvability of the Nonlinear Nonuniform Elliptic Equations with $L^1(D)$ Data

Farman Mamedov^{a,b}, Khayala Akhundova^{a,c}

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Azerbaijan State Oil and Industry University, Baku, Azerbaijan*

^c*Baku Higher Oil School (BHOS), SOCAR, Baku, Azerbaijan*

(e-mail: mammadovf1357@gmail.com, khayala.gasimova21@gmail.com)

In this paper, we investigate the existence of solutions to the Dirichlet problem associated with a nonlinear, non-uniformly degenerate elliptic equation of the form

$$-\operatorname{div} A(z, u, \nabla u) = f(z), \quad u|_{\partial D} = 0, \quad z \in D, \quad (1)$$

where the bounded domain $D \subset \mathbb{R}^N$. We assume that the right-hand side satisfies $f(z) \in L^1(D)$ (see, e.g., [2]), and the operator $A = (A_1(z, s, \zeta), \dots, A_N(z, s, \zeta)) : D \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$.

For a function $g : D \rightarrow \mathbb{R}$ where $z = (x, y)$, we define the weighted gradient as

$$\nabla_\omega g = (\omega^{1/p} \nabla_x g, \nabla_y g)$$

and its modulus as

$$|\nabla_\omega g| = (\omega^{2/p} |\nabla_x g|^2 + |\nabla_y g|^2)^{1/2}.$$

Furthermore, we assume that there exist positive constants c_1, c_2 such that the following structure conditions hold:

$$\begin{aligned} A(z, u, \nabla u) \cdot \nabla_z u &\geq c_1 |\nabla_\omega u|^p, \\ |A(z, u, \nabla u)| &\leq c_2 |\nabla_\omega u|^{p-1}, \quad p > 1. \end{aligned} \quad (2)$$

Let $q \geq p$ and $\omega : \mathbb{R}^N \rightarrow \mathbb{R}_+$ be a weight function in the Muckenhoupt A_p class such that, for every function $\varphi \in \operatorname{Lip}_0(D)$, the following Sobolev-type nonuniform gradient inequality holds (see [2]):

$$\|\varphi\|_{L^q(D)} \leq c \|\nabla_\omega \varphi\|_{L^p(D)}. \quad (3)$$

Theorem 1. *Let $D \subset \mathbb{R}^N$ be a bounded domain and $f \in L^1(D)$. Let ω be a function from the Muckenhoupt A_p -class such that inequality (3) holds. Then there exists a weak solution u to Problem (1)–(2) satisfying the inequality*

$$\|u\|_{WL^{\frac{q(p-1)}{p}}(D)} + \|\nabla_{\omega} u\|_{WL^{\frac{pq(p-1)}{p(q+1)-q}}(D)} \leq c_7 \|f\|_{L^1(D)}^{\frac{1}{p-1}},$$

in the weak topology of the Lebesgue spaces. The proofs are similar to [1].

References

- [1] L. Boccardo, T. Gallouet, *Nonlinear elliptic equations with right hand side measures*, In partial differential equations, 17, 1992, 641-655
- [2] F.I. Mamedov, *A Poincare's inequality with non-uniformly degenerating gradient*, Monatsh. fur Math. 194, 2021, 151-165.

THEORY OF PROBABILITY AND MATHEMATICAL STATISTICS

On convergence of branching processes with several types of particles and migration

Soltan Aliyev^a, Irade Ibadova^{a,b}

^a*Institute of Mathematics, Baku, Azerbaijan*

^b*Baku Business University, Baku, Azerbaijan*

(e-mail: soltanaliyev@yahoo.com, ibadovairade@yandex.ru)

From the point of view of the general theory of Markov processes, it is largely immaterial whether one considers a branching process with a single type of particle or with several particle types, since in the general case the process has a countable state space. However, from the standpoint of the specialized theorems of branching processes, this distinction is important. For example, in the case of several particle types, one can study the relationships among particles of different types, as well as classify the different types according to the possibilities of transformation from one type to another.

A detailed exposition of the theory of branching stochastic processes with several particle types is given in [1].

In this work the convergence of sequence of branching processes with several types of particles and emigration is investigated.

Let $\xi_n(\varepsilon) = (\xi_n^1(\varepsilon), \dots, \xi_n^d(\varepsilon))$, $n = 0, 1, 2, \dots$ be a sequence of discrete time branching processes with d -types of particle and generating function $F_\varepsilon(s) = (F_\varepsilon^1(s), \dots, F_\varepsilon^d(s))$, where ε series parametr, $s = (s_1, \dots, s_j)$, $F_\varepsilon^j(s) = M_j \left[s_1^{\xi_1^j(\varepsilon)} \dots s_d^{\xi_d^j(\varepsilon)} \right]$.

Here ξ_n^j is interpreted as a number of particles of j -type in the n -th generation, and M_j is conditional expectation on condition that at the beginning there was only one particle of j -th type.

Let $Q(s)$ be the generating function of emigrating particles.

The average number of j -type descendants from one particle of i -type denote by $a_{ij}(\varepsilon)$, i.e.

$$a_{ij}(\varepsilon) = M \left[\xi_1^j(\varepsilon) \mid \xi_0(\varepsilon) = e_i \right], \quad e_i = \left(\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{d-i} \right).$$

Denote

$$A(\varepsilon) = \|a_{ij}\|_{i,j=1}^d.$$

According to the behavior of matrix $A(\varepsilon)$ and some conditions on generating functions $F_\varepsilon(s)$, $Q_\varepsilon(s)$ the limit theorems on convergence of branching processes with many type of particles and emigration are obtained.

References

- [1] B.A. Sevastyanov, *Branching processes*, Izdat. Nauka, Moscow, 436 pp. (1971).

On the Probability Characteristics of the Ruin Moment of the Risk Process With Erlang(2) Distributed Claim Sizes

Ulviyya Idrisova^a, Rovshan Aliyev^b

^a*Azerbaijan State Oil and Industry University, Baku, Azerbaijan*

^b*Baku State University, Baku, Azerbaijan*

(e-mail: idrisova.ulviyya@asoiu.edu.az, rovshanaliyev@bsu.edu.az)

The study of ruin probabilities is a cornerstone of actuarial science, offering critical insights into the financial stability of insurance companies. Ruin occurs when an insurer reserve becomes insufficient to meet claims, making it essential to predict the likelihood of such an event.

Various models have been proposed to address this problem, with the classical risk model being a fundamental approach. According to Dickson and Waters [3]. The distribution of the time of ruin has been further explored by Lin and Willmot [2], who emphasized the importance of moments such as the surplus before ruin and the deficit in ruin.

Building on these foundational works, this paper explores the integro-differential equations governing the moments of the time of ruin. We focus on the special case where the claim sizes follow an Erlang(2) distribution, extending previous results through the use of Laplace transforms. The inclusion of a discount factor, as outlined in the work of Dickson [1], allows for a more refined analysis of ruin probabilities over finite time horizons.

Mathematical statement and main results. Let us consider the classical surplus process $U(t)$, $t \geq 0$, defined as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} \eta_i$$

$$N(t) = \max\{k : \sum_{i=1}^k \xi_i \leq t\},$$

where u is the initial surplus and c is the insurer's rate of premium income per unit of time.

Let us define the Laplace-Stieltjes transform of the ruin time:

$$\varphi(u, \delta) = E[e^{-\delta\tau_u} I(\tau_u < \infty)], \quad (1)$$

where δ is a non-negative parameter of the Laplace transform.

Theorem 1. Let a sequence of independent and identically distributed pairs of random variables

defined in a probability space $(\Omega, \mathcal{F}, P)$, such that ξ_i and η_i are independent random variables and let us assume that the claim sizes η follow an Erlang(2) distribution with density $f_\eta(t) = \alpha^2 t e^{-\alpha t}$, $\alpha > 0$, $t > 0$. Then, the following integral equation of the Laplace-Stieltjes transform of the $\varphi(u, \delta)$ will be derived

$$\begin{aligned} & \frac{\partial^3}{\partial u^3} \varphi(u, \delta) + \left(2\alpha - \frac{\lambda + \delta}{c}\right) \frac{\partial^2}{\partial u^2} \varphi(u, \delta) + \\ & + \left(\alpha^2 - 2\alpha \frac{\lambda + \delta}{c}\right) \frac{\partial}{\partial u} \varphi(u, \delta) - \frac{\alpha^2 \delta}{c} \varphi(u, \delta) = 0. \end{aligned} \quad (2)$$

Theorem 2. Let all the conditions of Theorem 1 be fulfilled. Then the solution of the differential equation (2) may be written as

$$\varphi(u, \delta) = \frac{\alpha^2}{(\alpha + \delta)^2} \frac{\left(\frac{\lambda + \delta}{c} - r_3(\delta)\right) e^{r_2(\delta)u} + \left(r_2(\delta) - \frac{\lambda + \delta}{c}\right) e^{r_3(\delta)u}}{r_2(\delta) - r_3(\delta)}, \quad (3)$$

where $r_1(\delta), r_2(\delta), r_3(\delta)$ are the roots of characteristic equation.

Differentiating $\varphi(u, \delta)$ once and twice with respect to δ and evaluating at $\delta = 0$, using the relation $E(\tau_u^n | \tau_u < \infty) = (-1)^n \frac{d^n}{d\delta^n} \varphi(u, \delta) |_{\delta=0}$.

Corollary. Let all the conditions of Theorem 2 be satisfied. Then the first and second moments of the ruin time are obtained as follows,

$$\begin{aligned} E(\tau_u | \tau_u < \infty) &= P_1(u)e^{r_2(0)u} + Q_1(u)e^{r_3(0)u}, \text{Var}(\tau_u | \tau_u < \infty) = \\ &= P_2(u)e^{r_2(0)u} + Q_2(u)e^{r_3(0)u} - (P_1(u)e^{r_2(0)u} + Q_1(u)e^{r_3(0)u})^2. \end{aligned}$$

where $P_1(u)$ and $Q_1(u)$ are linear functions of the initial surplus u , and $r_2(0), r_3(0)$ are the negative roots of the characteristic equation at $\delta = 0$. $P_2(u)$ and $Q_2(u)$ are quadratic functions of u derived from the second derivative of $\varphi(u, \delta)$. Full expressions for the coefficients are given in the paper.

Conclusion. This research provides a detailed analytical framework for determining the moments of time of ruin in a risk model where interarrival times follow an Erlang (2) distribution. By employing an integro-differential equation and Laplace transform techniques, we have successfully transformed the process into a solvable third-order linear homogeneous differential equation. The obtained differential equation allowed us to derive explicit expressions for the moments of the time of ruin. In particular, the first moment and the variance were calculated by differentiating the Laplace transform with respect to the transform techniques. These results offer practical tools for risk assessment and financial stability analysis in the insurance industry.

References

- [1] Dickson, D.C.M. Insurance Risk and Ruin. Cambridge University Press. (2005)
- [2] Lin, X., & Willmot, G. The Moments of the Time of Ruin, the Surplus Before Ruin, and the Deficit at Ruin. Insurance: Mathematics and Economics, 27, (2000) 19–44.
- [3] Dickson, D.C.M., & Waters, H.R. The Distribution of the Time to Ruin in the Classical Risk Model. ASTIN Bulletin, 32, (2002), 299–313.

Estimation of the Means of Branching Processes With Immigration

Hurshidjon Jumaqulov

Kokand State University, Kokand, Uzbekistan

(e-mail: xurshid81@gmail.com)

Let for each $n \in N$ $\{\xi_{k,j}^{(n)}, k, j \in N\}$ and $\{\varepsilon_k^{(n)}, k \in N\}$ are independent sequences of independent identical distributed, nonnegative integer - valued random variables with finite moment of second order. For each $n \in N$ we define the random process $\{X_k^{(n)}, k \geq 0\}$ by recursively

$$X_0^{(n)} = 0, \quad X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)}, \quad k \in N.$$

We assume that

$$m_n = E\xi_{1,1}^{(n)}, \quad \lambda_n = E\varepsilon_1^{(n)}, \quad \sigma_n^2 = \text{var} \xi_{1,1}^{(n)}, \quad b_n^2 = \text{var} \varepsilon_1^{(n)}$$

are finite for all $n \in N$. Define the step process $X_n(t)$ as an element of the Skorokhod space $D[0, T]$ by setting

$$X_n(t) = X_{[nt]}^{(n)}, \quad t \geq 0,$$

where $[a]$ denotes the integer part of the number a .

The following statement is taken place.

Theorem. Let the following conditions be met:

1) $m_n = 1 + \alpha d_n^{-1} + o(d_n^{-1})$ when $n \rightarrow \infty$, $\alpha > 0$, d_n - a sequence of positive numbers such that $nd_n^{-1} \rightarrow \infty$ when $n \rightarrow \infty$;

2) $\sigma_n^2 \rightarrow \sigma^2$ when $n \rightarrow \infty$;

3) there is a sequence of positive numbers c_n such that

$$c_n \rightarrow \infty \text{ and } c_n^{-1} \lambda_n \rightarrow \lambda, \quad (c_n d_n)^{-1} b_n^2 \rightarrow b^2 \text{ at } n \rightarrow \infty;$$

4) $E\left(\xi_{1,1}^{(n)} - m_n\right)^2 I\left(\left|\xi_{1,1}^{(n)} - m_n\right| > \theta d_n c_n^{\frac{1}{2}}\right) \rightarrow 0$ for $n \rightarrow \infty$ any $\theta > 0$;

5) $(d_n c_n)^{-1} E \left(\varepsilon_1^{(n)} - \lambda_n \right)^2 I \left(\left| \varepsilon_1^{(n)} - \lambda_n \right| > \theta d_n c_n^{\frac{1}{2}} \right) \rightarrow 0$ for $n \rightarrow \infty$ any $\theta > 0$. Then, when $n \rightarrow \infty$ weak convergence takes place

$$\left(m_n^{[nt]} d_n c_n^{\frac{1}{2}} \right)^{-1} (X_n(t) - EX_n(t)) \rightarrow W \left(2\alpha^{-1} (\alpha^{-1} \lambda \sigma^2 + b^2) \right)$$

W -standard Wiener process in space $D[0, T]$

References

- [1] Ispany M., Pap G., Van Zuijlen M.C.A. Fluctuation limits of branching processes with immigration and estimation of the means// Adv.Appl. Probab. 2005. -V.37. -P.523-538.
- [2] Khusanbaev Y.M. On asymptotics of branching processes with immigration. Discrete Mathematics and Applications. Vol. 28, Issue 1, 113-122 (2016).

Predicting Output Contractions in Emerging Economies: Probit Versus Random Forest

Ulvi Mammadzada

Institute of Mathematics, Baku, Azerbaijan
(e-mail: ulvimammadzada@yahoo.com)

The subject of this paper is a rule for assigning to each country and each year in a small macroeconomic panel a probability for the outcome that real output is contracting in the next year. The rationale for this is oil exporters with large oil sector share, and for which Azerbaijan serves as the case-in-point [3]. Let $y_{i,t+1} = 1$ if country i experienced a contraction in real GDP in year $t + 1$ and $y_{i,t+1} = 0$ otherwise. Given a vector $x_{i,t}$ of predictors observable in year t , we evaluate two methods of predicting the probability of $y_{i,t+1}$: one a parametric approach and the other based on decision trees, and we consider a

very simple question: does the more flexible of the two perform significantly better on such a dataset [1]

The first estimator is a probit model with a separate intercept for each country [5],

$$P(y_{i,t+1} = 1 \mid x_{i,t}) = \Phi(\alpha_i + \beta'x_{i,t}), \quad (1)$$

in which Φ is the standard normal distribution function and α_i captures fixed differences between countries that we do not observe. The second algorithm is a random forest [2] of 400 classification trees where class weights are balanced to place a higher weight on the minority class (contraction), where the weighting increases inversely with the frequency of occurrence so the learner does not simply default to the stable state always. Consequently, each economy is omitted in turn and is evaluated against one where training was conducted over the remaining five economies, in this way all evaluations are out of sample, including economies previously unused by the learner.

1995-2024 growth rates, as well as a range of other indicators, for Azerbaijan, Brazil, India, Indonesia, South Africa, and Türkiye, are from the World Bank. Once lags are added, the analysis covers 168 country-year observations, only 15 of which occurred immediately prior to a contraction. Azerbaijan is among the countries most exposed to external shocks, experiencing contraction events in 2011, 2016, and 2020 due to its close ties to oil price fluctuations. In the regression we consider, lagged growth, lagged inflation, the current account (expressed as a percentage of GDP), trade openness (as a percentage of GDP), exchange rate depreciation vis-a-vis the U.S. Dollar, and change in the percentage price of Brent oil are included as predictors.

We measure each model using the area under the receiver operating characteristics curve (AUC) [4]. In the within-sample estimate, the fixed-effects probit gives an AUC of 0.694 with a pseudo R-squared of only 0.073, and no coefficient is statistically significant (all $p > 0.35$). However, this within-sample performance overestimates the actual usefulness of the model since its fit is largely due to the country-specific intercepts and not to the economic predictors. Once these intercepts are dropped and the probit is validated out-of-sample, under the same leave-one-country-out strategy, its AUC drops to around 0.43 (i.e., below the chance level of 0.50), and its observed discrimination does not carry over to an unseen country. By contrast, out-of-sample (and conditioned on left-out countries), the random forest achieves an AUC of 0.667. In this like-for-like comparison on the out-of-sample evaluation set,

the forest performs better than the parametric model. The apparent victory of the probit was merely the result of comparing a within-sample performance measure to an out-of-sample measure. Using Youden's index for identifying the signaling cutoff, the probit correctly signals 5 true alarms against 46 false alarms (0.333 recall and 0.098 precision), and the forest signals 13 true alarms against 75 false alarms (0.867 recall and 0.148 precision). In the random forest, the most important variables (based on mean feature importance) are lagged growth (0.248), followed by exchange-rate depreciation (0.172) and the Brent crude oil price change (0.169).

Result 1. *On the 1995–2024 panel of six oil-exporting and emerging economies, under a common leave-one-country-out out-of-sample protocol, the probit classifier (1) loses its discriminatory power, while the random forest retains a modest but genuine out-of-sample signal,*

$$\text{AUC}_{\text{probit}}^{\text{oos}} \approx 0.43 < 0.50 < \text{AUC}_{\text{forest}}^{\text{oos}} \approx 0.667. \quad (2)$$

This points to two conclusions. The first is that when both models are compared on exactly the same out-of-sample exercise, the classical probit fails to improve upon the random forest. Its substantial within-sample discriminative power vanishes out of sample, as it is contingent upon country intercepts that do not generalize to a new economy. Out-of-sample AUC values around 0.667 are merely somewhat better than random, and both models sacrifice high recall in favour of low precision. The second conclusion is that the probit also generates useful, readily interpretable coefficients that can be communicated to policymakers. However, it no longer achieves comparable out-of-sample discriminative ability, meaning it should probably be used for explanation, not prediction, whereas the forest supplies the nonlinear out-of-sample prediction power. The dearth of trustworthy credit and fiscal data seems to be a greater impediment to advances in the field than the selection of prediction algorithm.

References

- [1] J. Beutel, S. List, G. von Schweinitz, *Does machine learning help us predict banking crises?*, Journal of Financial Stability, 2019, 45, 100693.
- [2] L. Breiman, *Random forests*, Machine Learning, 2001, 45(1), 5-32.

- [3] T.V. de V. Cavalcanti, K. Mohaddes, M. Raissi, *Commodity price volatility and the sources of growth*, Journal of Applied Econometrics, 2015, 30(6), 857-873.
- [4] T. Fawcett, *An introduction to ROC analysis*, Pattern Recognition Letters, 2006, 27(8), 861-874.
- [5] H. Nyberg, *Dynamic probit models and financial variables in recession forecasting*, Journal of Forecasting, 2010, 29(1-2), 215-230.

On the Strong Law of Large Numbers for the Family of First Passage Times of the Parabolic Boundary by the Markov Random Walk, Described by the Autoregressive Process $Ar(1)$

Fada Rahimov^{a,b}, Gulnara Baghirova^a, Vugar Khalilov^c,
Ulfat Mammadova^a

^a*Baku State University, Baku, Azerbaijan*

^b*Baku State University, Institute of Applied Mathematics, Baku, Azerbaijan*

^c*Institute of Mathematics, Baku, Azerbaijan*

(e-mail: ragimovf@rambler.ru, bagirova.gulnara.amir@bsu.edu.az)

We consider the first-order autoregressive process $AR(1)$ in the form

$$X_n - m = \beta (X_{n-1} - m) + \xi_n, \quad n \geq 1 \quad (1)$$

where $m, \beta \in (-\infty, \infty)$ are fixed constants and $\xi_n, \quad n \geq 1$ is a sequence of independent and identically distributed random variables (innovation) defined on the probability space $(O, \mathcal{F}, P)$.

Note that the process $AR(1)$ of the form (1) is considered in the works [1] [2] [3], where are studied tests of statistical hypothesis for unknown parameters m and β .

Let

$$L_n = \sum_{k=0}^n X_k, \quad n \geq 1$$

$L_n, n \geq 1$ is Markov random walk, described by sums of values of the process AR(1) in (1). The definition of a Markov random walk in general case is given in the work [1] (see also [2] [3]).

The Markov random walk of the form $L_n = \sum_{k=0}^n X_k, n \geq 1$ is generalization of the ordinary random walk, described by sums of independent and identically distributed random walk. In the case $\beta = 0$ we have for $n \geq 1$

$$L_n = \sum_{k=0}^n \eta_k = S_n, \eta_k = m + \xi_k$$

where $S_n, n \geq 1$ – ordinary random walk.

Boundary problems for ordinary random walks where studied in works [4] [5].

We consider the family of first passage time

$$\tau_a = \inf \{n \geq 1 : L_n > a\sqrt{n}\}, a > 0 \quad (2)$$

In the case $\beta = 0$ ($L_n = S_n$) there exist many results concerning the family of first passage times (2). The limit theorems (central limit theorem, law of large numbers, and strong law of large numbers) for the family of stopping times (2) in the case $\beta = 0$ were studied in [4], [5] (see also [2], [3]).

In present paper we prove the strong law of large numbers for the family $\{\tau_a, a \geq 0\}$ defined in (2) for $|\beta| < 1$

We assume that $E\xi_1 = 0, 0 < E\xi_1^2 = \sigma^2 < \infty, E|X_0 - m|^2 < \infty, |\beta| < 1$ and initial value of process AR(1) X_0 does not depend on the innovation $\{\xi_n\}$.

Under these conditions is well known process AR(1) is stationary and $E|X_n - m|^2 \rightarrow \frac{\sigma^2}{1-\beta^2}$ as $n \rightarrow \infty$ (see [4], [5]).

The following theorem is called the strong law of large numbers for the family of first passage times τ_a defined in (2).

Theorem. Let $|\beta| < 1, E\xi_1 = 0, D\xi_1 = \sigma^2 < \infty, E|X_n - m|^2 < \infty$ and

$$\frac{L_n}{n} \xrightarrow{a.s.} m > 0 \quad \text{as } n \rightarrow \infty \quad (3)$$

Then

$$\frac{\tau_a}{N_a} \xrightarrow{a.s.} \frac{1}{m} \quad \text{as } a \rightarrow \infty$$

where $N_a = \left(\frac{a}{m}\right)^2$

References

- [1] R.T. Aliyev, F.H. Rahimov, A. Farhadova. On the first passage time of the parabolic boundary by the Markov random walk. Communications in Statistics: Theory and Methods. 52(17), 6078–6087, 2022.
- [2] A. Gut. Stopped random walks. Limit theorems and applications. Springer, 1988 New York.
- [3] V.F. Melfi. Nonlinear Markov renewal theory with statistical applications. The Annals of Probability. 20(2), 753–771, 1992.
- [4] V.F. Melfi. Nonlinear renewal theory for Markov random walks. Stochastic Processes and their Applications. 54, 71–93, 1994.
- [5] D. Pollard. Convergence of Stochastic Processes. Springer, 1984, New-York.

CONTENTS

ALGEBRA, GEOMETRY AND TOPOLOGY

Sebuhi Abdullayev, Çiğdem Gündüz, Sadi Bayramov	
A Note on Perturbed Soft S-Metric Space	3
Furkan Arikan, Orhan Dişkaya	
An Extension of the Narayana Sequence to Non-Integer Indices	4
Marat Arslanov	
Computable and Non-Computable Real Numbers	4
Ali Babaev, Vafa Mammadova, Zumrud Panahova,	
Eminaga Mamedov	
Some Remarks on Uniform Algebras Valued Composition Operator	5
Rena Babayeva, Sabir Mirzoev, Eminaga Mamedov	
The Concept of Quantity and Equality in the Works of Nasir Al-Din	
Al-Tusi	8
Elmaga Gasymov, Arifa Kerimova	
Some Relations Between Trigonometric Functions and Algebraic	
Polynomials	11
Habil Fattayev	
Curvature Properties of the Fiberwise Berger type Deformed Sasaki	
Metric on the Coframe Bundle	14
Vagif Jabbarzade	
A k-Valued Discrete-Probability Logic of Event Formulas Without	
Combinatorial Explosion	16
Ljubiša Kočinac	
Topologies on locally finite graphs	19
Farkhod Mukhamadiev	
On Uniform Structures in the Superextension	19
Selma Özçağ, Haydar Saraf	
Topological Games in Bitopological Spaces: The Banach–Mazur Game	22
Kamala Veliyeva, Nuride Museyibova, Sadi Bayramov	
Homology Theory of Soft Sets in the Category of Shostak	
Topological Spaces	23

**COMPUTATIONAL MATHEMATICS AND
MATHEMATICAL MODELING**

Rahul Agarwal, Sumita Dahiya

A Polynomial Differential Quadrature–Crank–Nicolson Scheme for
a Reaction–Diffusion Dengue Epidemic Model 26

Kamil Aida-zade, Yegana Ashrafova

Computational Analysis of Fluid Flow in Complex Pipeline Networks 27

Konul Aliyeva, Khatira Babayeva

Analysis of Methods, Models and Algorithms of Mathematical
Modeling Used in Logistics Research 30

Rakhmatillo Alov, Dilfuza Nematova

Discrete Lyapunov Stability of a 3D Godunov-Type Scheme for
Symmetric Quasilinear Hyperbolic Systems with Dissipative
Boundary Conditions 33

Mirsaid Aripov, Makhmud Bobokandov

About the Solution of a Nonlinear Diffusion Equation in Non-
Divergence Form With a Source 36

Mirsaid Aripov, Firuza Kabiljanova, Maftuha Sayfullayeva

Mathematical Modeling of Nonlinear Biological Populations 38

Mirsaid Aripov Dilobar Nigmanova, Mohidil Rahmonova

Mathematical Modeling of Coupled Diffusion Systems
With Convective Transfer 42

Mirsaid Aripov, Dauletbay Utebaev, Sapura Kalmuratova

High-Accuracy Difference Schemes For A Multidimensional
Parabolic Equation 45

Sumita Dahiya, Suhana Chaudhary, Priyanka Yadav

Numerical Study of the Gilson–Pickering Equation for Nonlinear
Plasma Wave Dynamics 47

Serbun Ufuk Değer

Local Stability and the Emergence of Stable Two-Cycles in a Nonlinear
Difference System 48

Oybek Djabbarov, Mahbuba Samadova, Muzaffar Karimov

Qualitative Properties of Solutions to a Parabolic Heat Equation in
Nondivergence Form with Two Nonlinear Source Term 50

Vitaliy Goran

On the Numerical Stability of Branched Continued Fraction Expansions
of Some Ratios of Triple Hypergeometric Functions F_K 53

Allaoua Hemmak

Neuro-Metaheuristic Approache for Adaptive Vehicle Routing in
Urban Transport Networks 55

Ibtissam Sannaky, Imane El Berrai, Khalid Adnaoui

Dry Versus Wet Season Dynamics of Cystic Echinococcosis: Insights from
Stochastic Extinction Analysis and Deterministic Scenario Comparison 55

Akhmat Ikramov, Askhad Polatov,**Farrukhkhon Polatov, Wei Guo**

Convergence Analysis of Q4 and Q8 Elements for Elliptical Cavities
in Composite Plates 57

Arzu Mammadova, Konul Aliyeva

Applications of Functional Analysis in Physics 60

Alisher Matyakubov, Abrorjon Mamatov

Asymptotic Behavior of Self-Similar Solutions for Degenerate
Diffusion-Absorption Systems with Time-Dependent Coefficients
and Variable Density 63

Alisher Matyakubov, Jasurbek Salimov

On the Asymptotic Behaviour of Solutions to a Nonlinear Parabolic
System in Non-Divergence Form with Variable Density 66

Mahmoud Moustafa

Dynamical Analysis of Hantavirus Infection Model 68

Nargiza Nurmukhamedova, Nargiza Boltaeva

Sequential Fixed-Width Confidence Intervals for Right-Censored
Data via Averaged Stochastic Approximation 68

Aygun Omarova, Tofiq Orujov

Mathematical Modeling of Production Efficiency Under
Different Constraints 70

Askhad Polatov, Akhmat Ikramov, Sukhbatulla Pulatov,**Uchkunbek Adambaev**

Geometric Effects on Stress Localization in Fiber Composites 73

Anar Rahimov

On Numerical Solution to Inverse Problem of Restoring Special

Type Source of Parabolic Equation	76
Zafar Rakhmonov, Aziza Zaripova	
On the Properties of Solutions to a Multidimensional Nonlinear Heat Conduction Problem with Coupled Nonlinear Boundary Conditions	79
Alisher Rasulov, Gulnora Raimova	
Solution Semi-Linear Helmholtz Equation with a Quadratic Source term for Groundwater Flow	81
Azizaga Shakhverdiev, Yury Shestopalov	
Modeling the Instability of the Oil-Water Displacement Front During Multiphase Flow Based on a Polynomial Dynamical System	82
Sardorbek Murodov, Elvira Shakayeva, Chori Normurodov	
Numerical Solution of Two-Parameter Singularly Perturbed Boundary Value Problems Using the Spectral-Grid Method	85
Nietbay Uteuliev, Gafur Djaykov, Gulnaz Mamutova	
Explicit Inversion Formula and Regularization for Function Reconstruction from Circular Arc Integrals	88
Nietbay Uteuliev, Abat Seidullaev, Ilkhom Bektemirov	
Reconstruction Algorithms for a New Class of Weighted Integral Geometry Problems with Weight Function	90
Priyanka Yadav, Sumita Dahiya	
Scheme Based Comparison for Benjamin-Bona-Mahony-Burgers Equation Governing Flood Predictions	92
Zakir Zabidov, Khayala Gadirova	
Refinement of Low-Resolution Images with High-Resolution Images	94
Nasridin Zhabborov, Behzod Husenov, Abdurasul Narmanov	
A-Tomographic Transform and Main Properties	96

DIFFERENTIAL EQUATIONS AND OPTIMAL CONTROL

Vagif Abdullayev, Vugar Hashimov	
Feedback Control of Boundary Conditions and Sensor Motion During Rod Heating.....	98
Aygun Abdullayeva, Atif Namazov	
On Stabilization of Solutions to Benjamin-Bona-Mahony-Burgers Equation	101
Adalat Akhundov, Areste Habibova, Nahid Pashaev	
Inverse Problem for Burgers-Type Parabolic Equation	102

Nihan Aliyev, Ramin Zeynalov	
Investigation of the Solution of the Boundary Value Problem for a Linear, Homogeneous, Constant Coefficient Ordinary Differential Equation with Fractional Derivatives	105
Narmin Aliyeva, Yagub Aliyev	
Equivalence of Minimality Conditions	107
Rauf Amirov	
On The Inverse Nodal Problem For Diffusion Operator	110
Narmina Asgarova	
Second-Order Optimality Conditions for a Variational Problem with a Free Right Endpoint and Terminal Cos	112
Tahir Gadjev, Aybeniz Yagnaliyeva, Khalil Aliyev	
The Qualitative Property of Solutions Parabolic Non-Linear Equations	115
Elmaga Gasymov, Saddein Efendi	
Inversion Formula For Solving Parametric Problems With Irregular Boundary Conditions For Variable Coefficients Equations	118
Gülseren Çiçek	
On Duality of the Lagrange Problem with Differential Inclusions and multiple delays	121
Hamlet Guliyev, Tunzale Huseynova	
Optimal Control Problem for Second Order Hyperbolic Equations With Control in Coefficients	124
Hamlet Guliyev, Vusala Nasibzadeh	
Numerical Solution of the Problem of Determining the Initial Function In a Problem for a String Vibration Equation	126
Hamlet Guliyev, Hikmat Tagiyev	
An Optimal Control Problem for Unstable Parabolic Equation With a Nonlocal Boundary Condition	129
Sevinj Hadiyeva	
On a Problem for Hyperbolic Equations With Discontinuity Conditions ...	132
Asma Hanif	
Fractional optimal control analysis of Covid-19 and dengue fever co-infection model with Atangana-Baleanu derivative	134
Yousef Jafarzadeh	
Some New Results from Combining Taylor Series and Block Pulse	

Functions	136
Nazim Jafarov	
The Green Function for a Class of Second Order Parabolic Equations in Paraboloid Type Domains	137
Davron Juraev, Gulsum Aghayeva	
Optimal Control Problem for Matrix Factorizations of the Helmholtz Equation on the Plane	139
Baxtiyar Kadirkulov, Dilfuza Uzaqbaeva	
Direct and Inverse Problems for a Fourth-Order Mixed-Type Equation With Power Degeneration in a Three-Dimensional Domain	142
Adil Kahramanov	
Boundary Value Problems for Equations Not Resolved With Respect to the Highest Derivative	144
Mahbuba Kerimova	
On one Boundary Value Problem for Singularly Perturbed Quasilinear Differential Equation of Arbitrary Odd Order	146
Ikrombek Khajiev, Elbek Shobdarov	
Conditional Well-Posedness of an Initial-Boundary Value Problem for a Composite Pseudohyperbolic Equation	149
Elimhan Mahmudov, Gökhan Göksu	
Optimal Control of Parabolic Differential Inclusions Depending on First-Order Partial Derivatives	151
Eldar Mammadov	
On the Structure of the Numerical Range of a Multiparameter Problem under the Right Definiteness Condition	154
Akmal Mamatov	
On One Modular Maximin Control Problem	157
Elchin Mamedov, Aynur Hasanova	
On the Stabilization Of Generalized Solution for a Strongly Dissipative Hyperbolic Equation With a Nonlinear Source Term and Nonlinear Boundary Conditions	158
Elchin Mamedov, Aishan Mammadova	
On the Qualitative Properties of Generalized Solutions For The Heat Equation With a Strong Dissipative Term and Nonlinear Boundary Conditions	161

Famil Mammadov, Mesud Gahramanov	
The Cauchy Problem for a Class of Nonlinear Dissipative Hyperbolic Equations	164
Samin Malik	
Sufficient Conditions for a Minimum in a Nonsmooth Delayed Problem of the Calculus of Variations	166
Misir Mardanov, Telman Melikov, Aygun Isayeva	
Sufficient Conditions for Minimum in Delayed Variational Problems	169
Vagif Mastaliyev	
Application of the Method of Residues to the Solution of a Mixed Boundary Value Problem for a Class of Variable-Coefficient Differential Equations	172
Rashad Mastaliyev, Nicat Hasanli	
First Order Necessary Optimality Conditions in Stochastic Step Control Problem	175
Shemsiyye Muradova	
Investigation of Parabolic Fractional Integral Operators (with Rough Kernels) in Parabolic Local Generalized Morrey Spaces	177
Yelena Mustafayeva, Nihan Aliyev	
Necessary Conditions of Steklov Problem With Global Term in the Boundary Conditions for Laplace Equation	179
Malahat Najafova	
On a Necessary Optimality Condition for a Linear Nonlocal Discrete Control Problem with a Nondifferentiable Objective Functional	182
Gunay Rzayeva	
On the Triangular Representation of the Jost-Type Solution to the Perturbed Modified Mathieu Equation	185
Mahir Sabzaliev, Ilhama Sabzalieva	
On Asymptotics of Solution of Boundary Value Problem for Nonclassical Type Singularly Perturbed Quasilinear Differential Equation of Arbitrary Odd Order	188
Sevilay Demir Sağlama, Elimhan Mahmudov	
A Duality Framework for Optimal Control of Hyperbolic Differential Inclusions	190

Khayala Seyfullayeva	
Optimal Boundary Control for Plates in the Special Cases	193
Veli Shakhmurov	
Nonlocal Navier-Stokes Equations and Applications	196
Yagub Sharifov, Nazaket Mammadova	
Reduction of a Nonlocal Boundary Value Problem of a Second-Order Differential Equation to an Integral Equation	198
Brian Straughan	
Convection and Constitutive Memory: An Evolutionary Operator Viewpoint	201
Rafiq Tagiyev, Aytaj Mammadova	
On the Optimal Control Problem of a Second-Order Elliptic Equation with Periodic Boundary Conditions	202
Shakir Yusubov, Elimhan Mahmudov, Shikhi Yusubov	
On the Representation of Solutions to a Linear Hyperbolic Equation with Fractional Partial Derivatives in the Caputo Sense	205
Sergey Zelik	
Sharp Bounds for the Attractors Dimension for 2D Navier-Stokes Equations	208

FUNCTIONAL ANALYSIS AND THEORY OF OPERATORS

Yuliya Abbasova	
On the Absolute and Uniform Convergence of The Expansion in the Eigenfunctions one of a Boundary Value Problem for a Third-Order Differential Equation	209
Gulsum Aghayeva, Davron Juraev, Nurbala Suleymanov	
Boundary Value Problems for a Class of Fourth-Order Operator-Differential Equations on a Finite Interval	211
Shakhnoza Akhmatova	
Global Spectral Configurations of a Two-Particle Lattice Schrödinger Operator	214
Araz Aliev, Elshad Eyvazov	
Self-Adjointness of the Schrödinger Operator With Slowly Decaying Potential	216
Narmin Amanova, Rabil Amanov, Togrul Muradov	
An Oscillation Theorem for Solutions of Nonuniformly Degenerate Second-	

Order Elliptic–Parabolic Equations	218
Hamidulla Aslanov, Gunel Eyvazli, Kamilla Alimardanova	
Investigation of Resolvent of Operator Generated by Differential Expressions With Operator Coefficients in Hilbert space.....	220
Özlem Baksi, Kemal Demir, Baharchin Akhmadli	
The Strong Solvability of a Nonlocal Boundary Value Problem for the Poisson Equation in a Rectangle	223
Bilal Bilalov, Ümit Ildiz, Yonca Sezer	
Solvability of a Nonlocal Problem in Orlicz-Sobolev Spaces	224
Mohammed Benharat	
Solvability of a class of a linear fourth-order boundary value problems in Hilbert spaces	225
Alper Erdem	
Semi Cauchy Variants of Alpha Convergence and Exhaustiveness with Korovkin-Type Theorems	227
Alper Erdem, Engin Furat	
A Note On Convergence Notions Invariant Under Uniform Conjugacy	228
Nigar Gadirli	
On Asymptotics of the Green Function Cauchy Problem for Parabolic Equation Corresponding Higher Order Operator-Differential Equation in Hilbert Space	229
Telman Gasymov, Ilkin Feyzullayev, Amiragha Shikhammadov	
On the Strong Solvability of a One-Dimensional Parabolic Problem With a Non-Classical Boundary Condition	231
Elchin Ibadov	
On Convergence of the Spectral Expansion in Root Vector Functions of Discontinuous Schrödinger Operator With Matrix Potential.....	234
Migdad Ismailov, Chinara Hasanova	
Grand Classes of Hardy and the Cauchy Formula	237
Zaur Kasumov, Nigar Ahmedzade, Tarlan Garayev	
Some Properties of X-valued Poisson Integrals in Bochner Spaces	239
Seda Kızılbudak Çalışkan, Sümeyye Gica, Baharchin Akhmadli	
Uniqueness and Existence of One Nonlocal Problem in Orlicz-Sobolev Space	242

Ahmet Köprücü, Uğur Değer

Degree of Approximation for Harmonically Modulated Functions in the Hölder Class.....243

Veli Kurbanov, Leyla Buksayeva

Estimating the Local Equiconvergence Rate for the one –Dimensional Schrödinger Operator With Summable Potential 244

Sema Metina, Uğur Değer

Roughness Behaviors of Fractal Interpolation Functions via Ces‘aro Mean . 247

Heybetkulu Mustafayev

Mean Ergodic Theorem for Probability Measures 248

Sabina Sadigova, Vafa Alili, Asmar Mirzabalayeva

Schauder-Type Estimates for Higher Order Elliptic Equations in Grand Sobolev Spaces 249

Ekrem Savaş

On generalized almost statistical convergence of weight g in partial metric spaces 250

THEORY OF FUNCTIONS**Fahreddin Abdullayev, Meerim Imashkyzy**

On the Behavior of $m - th$ ($m \geq 1$) Derivatives of Algebraic Polynomials in Unbounded Domains of the Complex Plane 251

Fahreddin Abdullayev, Viktor Savchuk

Lower estimates for the best approximation of continuous holomorphic functions 254

Yagub Aliyev, Zamina Guliyeva

On the Generalization of Sierpinski’s Inequality 255

Roman Dmytryshyn

On the Approximation of Special Functions by Branched Continued Fractions 256

Marta Dmytryshyn, Volodymyr Hladun

On the Stability to Perturbations of S -fractions 257

Behzod Husenov

The Geometry of $A(z)$ -Analytic Spaces and the Bergman Reproducing Kernel 258

Vugar Ismailov

Approximation Properties of Feedforward Neural Networks on Topological Spaces.....261

Daniyal Israfilov

Approximation Theorems in Variable Exponent Smirnov Classes 263

Kirill Kopotun

Comonotone Approximation by Periodic Splines of Minimal Defect 266

Ivan Nyzhnyk

On the Analytical Extension of Some Ratios of Triple Hypergeometric Functions F_M 267

Viktor Savchuk, Maryna Savchuk

Coefficient Estimates for Bounded Holomorphic Functions 269

Niyaz Tokmagambetov

Pitt-Type Inequalities for General Monotone Functions 270

NON-HARMONIC ANALYSIS**Aytekin Abdullayeva, Aynur Mammadova, Lale Aliyeva**

Asymptotic Properties of Bivariate Bernstein-Chlodowsky Polynomials on a Triangle 271

Rovshan Bandaliyev, Aytekin Abdullayeva, Dunya Aliyeva

On the Boundedness of One-Dimensional Hausdorff Operators in Orlicz Spaces.....274

Rovshan Bandaliyev, Vugar Khalilov, Khalida Nasirova

On Kolmogorov-Riesz Compactness Theorem in Quasi-Grand Lebesgue Space 277

Aynur Huseynli

On the Integrability of the Boundary Values of Harmonically Conjugate Functions 280

Elman Ibrahimov, Aynura Binnatova, Saadat Jafarova

Gegenbauer Fractional Integral Operator and its Commutators on Gegenbauer-Orlicz Spaces 282

Rahim Rzaev, Lale Aliyeva, Surayye Isayeva

Description of the Generalized Besov Spaces in Terms of Mean Oscillation .285

PROBLEMS OF MECHANICS

Elkhan Abbasov, Nurlana Agayeva, Aynur Darishova Simulation of Stationary Annular Flow of a Gas-Liquid Mixture in a Pipe, Considering Radial Viscosity Variation of the Gas Cushion.....	288
Elkhan Abbasov, Aynur Darishova Dynamics of Gas-Liquid Mixture Flow in a Pipe, Taking Into Account Phase Transitions and Heat Exchange With the Environment	291
Elkhan Abbasov, Serkar Imamaliyev Modeling the Dynamics of a Gas-Liquid Mixture Flow in a Pipe, Taking Into Account Phase Transformations and Pipe Wall Deformation	294
Surxay Akbarov, Emin Bagirov, Fuad Huseynov, Arzu Abbasova On the Wave Propagation in the Eccentric Hollow Cylinder Containing an Inviscid Compressibl Fluid	297
Ramil Bakhtizin, Geylani Panahov, Fuad Khudaverdiyev On Some a Spects of Mathematical Modeling of Heat Conduction in Nonequilibrium Multicomponent Media	299
Sona Farajova, Yusif Sevdimaliyev Dynamics of an Initially Stressed Eccentric Hollow Cylinder	302
Elshan Hashimov, Bayram Ibrahimov, Marko Radovanovic, Aziz Talibov, Valeh Bakhshali, Ismayil Ismayil Assessment of the Technical Condition of Mechanical Systems Using Digital Twin and Artificial Intelligence Technologies	305
Aynura Khudiyeva Dynamics of Flexural-Gravity-Capillary Waves in Floating Viscoelastic Plates under Stochastic Loading	308
Bakhtiyor Khuzhayorov, Tursunpulat Dzhiyanov Inverse Problem For Colloidal Particle Transport In A Two-Zone Porous Medium With Different Properties.....	311
Hijran Mammadova Corrosion-Induced Failure of a Circular Concentric Plate Under Internal Pressure	315
Mehriban Mammadova Taking Into Account the Influence of Groundwater on the Durability of a Vertical Well	318

Ibrakhim Mirzev, Sultan Gaynazarov	
Mathematical Modeling of Oblique Impact of Elastic Waves on a Parallelepiped With Transparent Borders	320
Mahir Mehdiyev, Sahib Aliyev	
Three -Dimensional Dynamics of Non-Axisymmetric Moving Load Acting in the Interior of the Bi-Layered Hollow Cylinder	322
Irada Mirzazadeh, Gulchin Abdullayeva	
Data-Driven Long-Term Monitoring After Carbon Monoxide Intoxication: A Framework for Clinical Decision Support	326
Ayten Muradova	
Cyclic Torsion of a Viscoelastoplastic Bar: Deformation and Long-term Strength Prediction	329
Nigar Nagiyeva	
Theoretical Investigation of Elastic-Plastic Cyclic Torsion and Fatigue Features Minimum Cross-Section Rectangular Torque Structural Elements .	330
Geylani Panahov, Eldar Abbasov, Afat Yuzbashiyeva,	
Ibrahim Mamedov	
Electrokinetic Effects During the Displacement of Miscible Electrolytes in a Porous Medium	331
Geylani Panahov, Nazim Valiyev, Eldar Abbasov, Babek Sultanov	
Assessment of Near-Wellbore Formation Stability Under Hydromechanical and Thermochemical Effects	334
Nazila Rassoulova, Mubariz Rassoulov, Tehmine Mahmudzade	
Solution of Lamb's Problem for an Elastic Half-Space	337
Vusala Rzayeva	
Vibrations of a Cylindrical Shell Stiffened With Inhomogeneous Rods and Rings	339
Tebritz Shamiev, Ruhiyye Zamanova, Rena Ismailova	
Stability of the Plane Form of Bending of Thin-Walled Rods with Regard to Bimodularity	340
Famil Seyfullaev, Alikhan Garayev	
Influence of Effect of a Sliding Friction on the Characteristic of Natural Frequencies of the Longitudinally Supported Cylindrical Envelopes With Filler at an Axial Compression	343

NONLINEAR PROBLEMS OF MATHEMATICAL PHYSICS**Attila Aşkar**

A Promenade in the Nonlinear World 347

Shirmail Bagirov

Absence of Global Solutions in an Infinite Domain for A System of Second-Order Semilinear Equations With Fractional Time Derivatives 347

Michel Chipot

On Some Liouville Theorems for p-Laplace Type Operators 350

Varga Kalantarov

Stabilization of Damped Nonlinear Wave Equations by Controllers Depending on Finitely Many Parameters 350

Farman Mamedov, Khayala AkhundovaOn the Weak Solvability of the Nonlinear Nonuniform Elliptic Equations with $L^1(D)$ Data 351**THEORY OF PROBABILITY AND MATHEMATICAL STATISTICS****Soltan Aliyev, Irade Ibadova**

On convergence of branching processes with several types of particles and migration 353

Ulviyya Idrisova, Rovshan Aliyev

On the Probability Characteristics of the Ruin Moment of the Risk Process With Erlang(2) Distributed Claim Sizes 354

Hurshidjon Jumaqulov

Estimation of the Means of Branching Processes With Immigration 357

Ulvi Mammadzada

Predicting Output Contractions in Emerging Economies: Probit Versus Random Forest 358

Fada Rahimov, Gulnara Baghirova, Vugar Khalilov,**Ulfat Mammadova**On the Strong Law of Large Numbers for the Family of First Passage Times of the Parabolic Boundary by the Markov Random Walk, Described by the Autoregressive Process $Ar(1)$ 361

